LICS 2021 “Inspirational Lecture”: Pseudo-Boolean Solving

Jakob Nordström

University of Copenhagen and Lund University

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Pseudo-Boolean?

Pseudo-Boolean (PB) function: $f : \{0,1\}^n \to \mathbb{R}$

Studied since 1960s in operations research and 0-1 integer linear programming [BH02]

Such a function f can always be represented as polynomial

Restriction for this lecture: f represented as linear form

Many problems expressible as optimizing value of linear pseudo-Boolean function under linear pseudo-Boolean constraints

Pseudo-Boolean vs. SAT

- PB format richer than conjunctive normal form (CNF)

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
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- Yet close enough to SAT to benefit from SAT solving advances
- Also possible synergies with 0-1 integer linear programming (ILP)

Outline of Lecture

1. Preliminaries
   - Pseudo-Boolean Constraints
   - Pseudo-Boolean Solving and Optimization
2. Conflict-Driven Pseudo-Boolean Solving
   - The Conflict-Driven Paradigm
   - Pseudo-Boolean Reasoning Using Saturation
   - Pseudo-Boolean Reasoning Using Division
3. Going Beyond the State of the Art?
   - Other Pseudo-Boolean Reasoning Rules
   - Challenges
   - Some Further References

Pseudo-Boolean Constraints and Normalized Form

For us, pseudo-Boolean constraints are always 0-1 integer linear constraints

$$\sum_i a_i \ell_i \bowtie A$$

- $\bowtie \in \{\geq, \leq, =, >, <\}$
- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Convenient to use normalized form [Bar95] (without loss of generality)

$$\sum_i a_i \ell_i \geq A$$

- constraint always greater-than-or-equal
- $a_i, A \in \mathbb{N}$
- $A = \mathit{deg}(\sum_i a_i \ell_i \geq A)$ referred to as degree (of falsity)

Some Types of Pseudo-Boolean Constraints

1. Clauses are pseudo-Boolean constraints

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3. General constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Conversion to Normalized Form: Example

Normalized form used for convenience and without loss of generality

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3. Replace $-\ell$ by $-(1-\overline{\ell})$ [where we define $\overline{\overline{x}} \doteq x$]

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4. Replace "$=$" by two inequalities "$\geq$" and "$\leq$"

Formulas, Decision Problems, and Optimization Problems

Pseudo-Boolean (PB) formula

Conjunction of pseudo-Boolean constraints

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Find satisfying assignment to F minimizing objective function $\sum_i w_i \ell_i$

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This lecture:

- Focus on pseudo-Boolean solving
- But not hard to extend to (simple) optimization algorithm

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Note: In all of these examples, the problem is to

- optimize a linear function
- subject to a CNF formula (all constraints are clausal)

Already expressive framework!

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- Assign values to variables (in some smart way)
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CDCL Main Loop Pseudocode

CDCL(F)

$\mathcal{D} \leftarrow F$; `// initialize clause database to contain formula`
$\rho \leftarrow \emptyset$; `// initialize assignment trail to empty`
forever do
  if ρ *falsifies some clause* $C \in \mathcal{D}$ **then**
    $A \leftarrow$ analyzeConflict$(\mathcal{D}, \rho, C)$;
    if $A = \perp$ **then** output `UNSATISFIABLE` and exit;
    else
      add A to $\mathcal{D}$ and backjump by shrinking ρ ;
  else if *exists clause* $C \in \mathcal{D}$ *unit propagating* x *to* $b \in \{0, 1\}$ *under* ρ **then**
    add propagated assignment $x \stackrel{D}{=} b$ to ρ ;
  else if *time to restart* **then** $\rho \leftarrow \emptyset$;
  else if *time for clause database reduction* **then**
    erase (roughly) half of learned clauses in $\mathcal{D} \setminus F$ from $\mathcal{D}$
  else if *all variables assigned* **then** output `SATISFIABLE` and exit;
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Conflict Analysis Pseudocode

analyzeConflict$(\mathcal{D}, \rho, C_{\mathrm{confl}})$

$C_{\mathrm{learn}} \leftarrow C_{\mathrm{confl}}$;
while C_{learn} *not UIP clause* **and** $C_{\mathrm{learn}} \neq \bot$ **do**
    $\ell \leftarrow$ literal assigned last on trail ρ;
    if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
        $C_{\mathrm{reason}} \leftarrow \mathrm{reason}(\ell, \rho, \mathcal{D})$;
        $C_{\mathrm{learn}} \leftarrow \mathrm{resolve}(C_{\mathrm{learn}}, C_{\mathrm{reason}})$;
    $\rho \leftarrow \rho \setminus \{\ell\}$;
return C_{learn};

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Conversion to disjunctive clauses

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Native reasoning with pseudo-Boolean constraints

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- Variable assignments
  1. Always propagate forced assignment if possible
  2. Otherwise make assignment using decision heuristic
- At conflict
  1. Do conflict analysis to derive new constraint
  2. Add new constraint to constraint database
  3. Backjump by rolling back decisions so that learned constraint propagates asserting literal (flipping it to opposite value)

Propagation, Conflict, and Slack

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Represent as $\rho = \{$(ordered) set of literals assigned true$\}$

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$\{\overline{x}_5, \overline{x}_4\}$	3	propagation doesn't change slack
$\{\overline{x}_5, \overline{x}_4, \overline{x}_3, x_2\}$	-2	conflict (slack < 0)

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Represent as $\rho = \{$(ordered) set of literals assigned true$\}$

Slack measures how far ρ is from falsifying $\sum_i a_i \ell_i \geq A$

$$slack(\textstyle\sum_i a_i \ell_i \geq A; \rho) = \displaystyle\sum_{\ell_i \text{ not falsified by } \rho} a_i - A$$

Consider $C \doteq x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

ρ	$slack(C; \rho)$	comment
$\{\}$	8	
$\{\overline{x}_5\}$	3	propagates $\overline{x}_4$ (coefficient $>$ slack)
$\{\overline{x}_5, \overline{x}_4\}$	3	propagation doesn't change slack
$\{\overline{x}_5, \overline{x}_4, \overline{x}_3, x_2\}$	-2	conflict (slack < 0)

Note: constraint can be conflicting though not all variables assigned

Conflict Analysis Invariant

Consider example CDCL conflict analysis from SAT solving lecture

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

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Assignment "left on trail" always falsifies derived clause

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Assignment "left on trail" always falsifies derived clause

$\overline{y} \lor \overline{z}$ falsified by trail $\rho = \{\overline{p}, \overline{u}, \overline{q}, r, w, \overline{x}, y, z\}$

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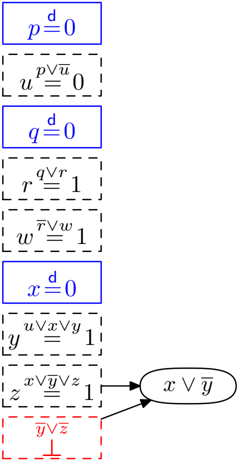

$x \lor \overline{y}$ falsified by trail $\rho' = \{\overline{p}, \overline{u}, \overline{q}, r, w, \overline{x}, y\}$

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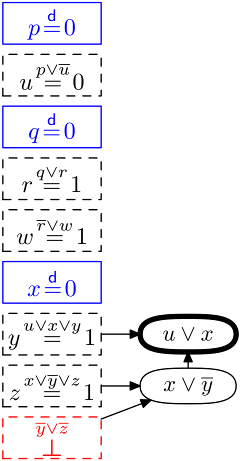

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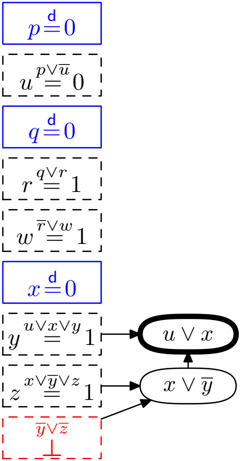
Conflict Analysis Invariant

Consider example CDCL conflict analysis from SAT solving lecture

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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$\Rightarrow$ derived clause "explains" conflict

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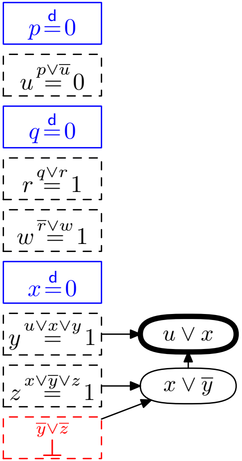

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Terminate analysis when explanation "looks nice"

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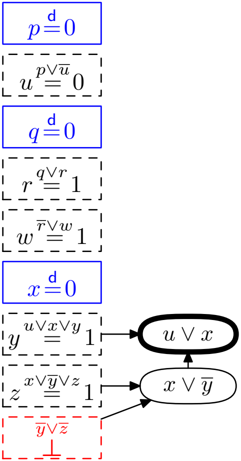

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Assignment "left on trail" always falsifies derived clause

$\Rightarrow$ derived clause "explains" conflict

Terminate analysis when explanation "looks nice"

Namely: after backjump, some variable guaranteed to flip

Generalized Resolution

Can mimic resolution step

$$\frac{x \vee \overline{y} \vee z \qquad \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

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$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

by adding clauses as pseudo-Boolean constraints

$$\frac{x + \overline{y} + z \geq 1 \qquad \overline{y} + \overline{z} \geq 1}{x + 2\overline{y} \geq 1}$$

(Recall $z + \overline{z} = 1$)

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(Recall $z + \overline{z} = 1$)

Generalized resolution rule (from [Hoo88, Hoo92])
Positive linear combination so that some variable cancels

$$\frac{a_1 x_1 + \sum_{i \geq 2} a_i \ell_i \geq A \qquad b_1 \overline{x}_1 + \sum_{i \geq 2} b_i \ell_i \geq B}{\sum_{i \geq 2} \left(\frac{c}{a_1} a_i + \frac{c}{b_1} b_i\right) \ell_i \geq \frac{c}{a_1} A + \frac{c}{b_1} B - c} \; [c = \operatorname{lcm}(a_1, b_1)]$$

Saturation

Actually, not quite the right constraint in mimicking of resolution

$$\frac{x + \overline{y} + z \geq 1 \qquad \overline{y} + \overline{z} \geq 1}{x + 2\overline{y} \geq 1}$$

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Saturation rule

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min\{a_i, A\} \cdot \ell_i \geq A}$$

Sound over integers, not over reals (need such rules for SAT solving)

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Sound over integers, not over reals (need such rules for SAT solving)

[Generalized resolution as defined in [Hoo88, Hoo92] includes fix above, but convenient here to make the two separate steps explicit]

Analyze Conflict with Generalized Resolution + Saturation!

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
$$C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$$

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Trail $\rho = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1, x_3 \stackrel{C_1}{=} 1\}$ $\Rightarrow$ Conflict with C_2

(Note: same constraint can propagate several times!)

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- Resolve reason$(x_3, \rho) = C_1$ with C_2 over x_3 to get resolve(C_1, C_2, x_3)

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- Applying saturate$(x_4 \geq 1)$ does nothing
- Non-negative slack w.r.t. $\rho' = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1\}$ — not conflicting!

What Went Wrong? And What to Do About It?

Accident report

- Generalized resolution sound over the reals
- Given $\rho' = \{x_1 = 0, x_2 = 1\}$, over the reals have
  - $C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$ propagates $x_3 \geq \frac{1}{2}$
  - $C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$ satisfied by $x_3 \leq \frac{1}{2}$
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- So after resolving away x_3 no conflict left!

Remedial action

- Strengthen propagation to $x_3 \geq 1$ also over the reals
- I.e., want reason C with $slack(C; \rho') = 0$
- **Fix** (non-obvious): Apply weakening

$$\text{weaken}\left(\textstyle\sum_i a_i \ell_i \geq A, \ell_j\right) \doteq \textstyle\sum_{i \neq j} a_i \ell_i \geq A - a_j$$

  to reason constraint and then saturate
- Approach in [CK05] (goes back to observations in [Wil76])

Try to Reduce the Reason Constraint

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
$$C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$$

Trail $\rho = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1, x_3 \stackrel{C_1}{=} 1\}$ $\Rightarrow$ Conflict with C_2

Let's try to

1. Weaken reason on non-falsified literal (but not last propagated)
2. Saturate weakened constraint
3. Resolve with conflicting constraint over propagated literal

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$$\text{resolve } x_3 \dfrac{\text{saturate } \dfrac{\text{weaken } x_2 \dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 + x_4 \geq 2}}{2x_1 + 2x_3 + x_4 \geq 2} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 + x_4 \geq 1}$$

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$$\dfrac{\dfrac{\dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 + x_4 \geq 2}\text{weaken } x_2}{2x_1 + 2x_3 + x_4 \geq 2}\text{saturate} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 + x_4 \geq 1}\text{resolve } x_3$$

Bummer! Still non-negative slack — not conflicting

Try Again to Reduce the Reason Constraint...

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
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$$\dfrac{\text{resolve } x_3 \dfrac{\text{saturate } \dfrac{\text{weaken } \{x_2, x_4\} \dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 \geq 1}}{x_1 + x_3 \geq 1}}{} \quad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 \geq 1}$$

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weaken $\{x_2, x_4\}$ $\dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 \geq 1}$

saturate $\dfrac{2x_1 + 2x_3 \geq 1}{x_1 + x_3 \geq 1}$

resolve x_3 $\dfrac{x_1 + x_3 \geq 1 \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 \geq 1}$

Negative slack — conflicting!

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Negative slack — conflicting!

Backjump propagates to conflict without solver making any decisions

Done! Next conflict analysis will derive contradiction

(Or, in practice, terminate immediately at conflict without decisions)

Reason Reduction Using Saturation [CK05]

reduceSat($C_{\text{reason}}, C_{\text{learn}}, \ell, \rho$)

while slack(resolve($C_{\text{learn}}, C_{\text{reason}}, \ell$); ρ) ≥ 0 **do**
    $\ell' \leftarrow$ literal in $C_{\text{reason}} \setminus \{\ell\}$ not falsified by ρ;
    $C_{\text{reason}} \leftarrow$ saturate(weaken(C_{reason}, ℓ'));
return C_{reason};

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Why does this work?

- Slack is subadditive

$$\mathit{slack}(c \cdot C + d \cdot D; \rho) \leq c \cdot \mathit{slack}(C; \rho) + d \cdot \mathit{slack}(D; \rho)$$

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$$\mathit{slack}(c \cdot C + d \cdot D; \rho) \leq c \cdot \mathit{slack}(C; \rho) + d \cdot \mathit{slack}(D; \rho)$$

- By invariant have $\mathit{slack}(C_{\mathrm{learn}}; \rho) < 0$

Reason Reduction Using Saturation [CK05]

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while $\mathit{slack}(\text{resolve}(C_{\text{learn}}, C_{\text{reason}}, \ell); \rho) \geq 0$ **do**
  $\ell' \leftarrow$ literal in $C_{\text{reason}} \setminus \{\ell\}$ not falsified by ρ;
  $C_{\text{reason}} \leftarrow \text{saturate}(\text{weaken}(C_{\text{reason}}, \ell'))$;
return C_{reason};

Why does this work?

- Slack is subadditive

$$\mathit{slack}(c \cdot C + d \cdot D; \rho) \leq c \cdot \mathit{slack}(C; \rho) + d \cdot \mathit{slack}(D; \rho)$$

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- Weakening leaves $\mathit{slack}(C_{\text{reason}}; \rho)$ unchanged
- Saturation decreases slack — hit 0 when max #literals weakened

Pseudo-Boolean Conflict Analysis Pseudocode

analyzePBconflict$(\mathcal{D}, \rho, C_{\text{confl}})$

$C_{\text{learn}} \leftarrow C_{\text{confl}}$;
while C_{learn} *not asserting* **and** $C_{\text{learn}} \neq \bot$ **do**
  $\ell \leftarrow$ literal assigned last on trail ρ;
  if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
    $C_{\text{reason}} \leftarrow \text{reason}(\ell, \rho, \mathcal{D})$;
    $C_{\text{reason}} \leftarrow \text{reduceSat}(C_{\text{reason}}, C_{\text{learn}}, \ell, \rho)$;
    $C_{\text{learn}} \leftarrow \text{resolve}(C_{\text{learn}}, C_{\text{reason}}, \ell)$;
    $C_{\text{learn}} \leftarrow \text{saturate}(C_{\text{learn}})$;
  $\rho \leftarrow \rho \setminus \{\ell\}$;
return C_{learn};

Reduction of reason new compared to CDCL — otherwise the same
Essentially conflict analysis used in SAT4J [LP10]

Some Problems Compared to CDCL

- Compared to clauses harder to detect propagation for constraints like

$$\sum_{i=1}^{n} x_i \geq n - 1$$

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- Generalized resolution for general pseudo-Boolean constraints
$\Rightarrow$ lots of lcm computations
$\Rightarrow$ coefficient sizes can explode (expensive arithmetic)
- For CNF inputs, degenerates to resolution!
$\Rightarrow$ CDCL but with super-expensive data structures

The Cutting Planes Proof System

Cutting planes as defined in theory literature [CCT87] doesn't use saturation but instead division (a.k.a. Chvátal-Gomory cut)

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B}$$

Division $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil}$$

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- Can division yield stronger conflict analysis?

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- Cutting planes with division implicationally complete
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- Can division yield stronger conflict analysis?
  (Used for integer linear programming in CutSat [JdM13])

Using Division to Reduce the Reason

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
$$C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$$

Trail $\rho = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1, x_3 \stackrel{C_1}{=} 1\}$ $\Rightarrow$ Conflict with C_2

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$$\dfrac{\text{resolve } x_3 \quad \dfrac{\text{divide by } 2 \quad \dfrac{\text{weaken } x_4 \quad 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_2 + 2x_3 \geq 3}}{x_1 + x_2 + x_3 \geq 2} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{0 \geq 1}$$

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Terminate immediately!

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reduceDiv($C_{\text{reason}}, C_{\text{learn}}, \ell, \rho$)

$c \leftarrow \mathit{coeff}(C_{\text{reason}}, \ell)$;
while slack(resolve(C_{learn}, divide(C_{reason}, c), ℓ); ρ) ≥ 0 **do**
    $\ell_j \leftarrow$ literal in $C_{\text{reason}} \setminus \{\ell\}$ such that $\overline{\ell}_j \notin \rho$ and $c \nmid \mathit{coeff}(C, \ell_j)$;
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- Slack same after weakening $\Rightarrow$ always $0 \leq \mathit{slack}(C_{\text{reason}}; \rho) < c$
- After max #weakenings have $0 \leq \mathit{slack}(\text{divide}(C_{\text{reason}}, c); \rho) < 1$

Division vs. Saturation

- Higher conflict speed when PB reasoning doesn't help [EN18]
- Seems to perform better when PB reasoning crucial [EGNV18]
- Keeps coefficients small — can (often) do fixed-precision arithmetic
- But SAT4J still better for some circuit verification problems [LBD+20]
- And still equally hard to detect propagation
- Also, still degenerates to resolution for CNF inputs
- Sometimes very poor performance even on infeasible 0-1 LPs!

Other PB Rules I: Cardinality Constraint Reduction

Given PB constraint

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Cardinality constraint reduction rule

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_{i\,:\,a_i>0} \ell_i \geq T} \quad T = \min\{|I| \,:\, I \subseteq [n], \sum_{i \in I} a_i \geq A\}$$

Can be simulated with weakening + division

Other PB Rules II: Strengthening

Strengthening by example:

- Set $x = 0$ and propagate on constraints

$$x + y \geq 1 \qquad x + z \geq 1 \qquad y + z \geq 1$$

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Strengthening rule (imported by [DG02] from operations research)

- Suppose $\ell = 0 \Rightarrow \sum_i a_i \ell_i \geq A$ oversatisfied by amount K
- Then can deduce $K\ell + \sum_i a_i \ell_i \geq A + K$

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In theory, can recover from bad encodings (e.g., CNF)
In practice, seems inefficient and hard to get to work. . .

Other PB Rules III: "Fusion Resolution"

Suppose have constraints

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But only get from resolution + saturation

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"Fusion resolution" [Goc17]

$$\frac{a\ell + \sum_i b_i\ell_i \geq B \qquad a\overline{\ell} + \sum_i b_i\ell_i \geq B'}{\sum_i b_i\ell_i \geq \min\{B, B'\}}$$

No obvious way for cutting planes to immediately derive this
Shows up in some tricky benchmarks in [EGNV18]

Some PB Solving Challenges I: Input Format

1. CNF: PB solvers degenerate to CDCL for CNF inputs — how to harness power of cutting planes in this setting?
   - Cardinality constraint detection proposed as preprocessing [BLLM14] or inprocessing [EN20]
   - Not yet competitive in practice

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   - Work on addressing this in [DGN21]

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2. Linear programming: Sometimes very poor performance even on infeasible 0-1 LPs!
   - Unclear why — very easy for cutting planes in theory
   - Work on addressing this in [DGN21]
3. Preprocessing/presolving: Important in SAT solving and integer linear programming, but not done in PB solvers — why?
   - Follow up on preliminary work on PB preprocessing in [MLM09]?
   - Use presolver PaPILO [PaP] from mixed integer linear programming (MIP) solver SCIP [SCI]?

Some PB Solving Challenges II: Conflict Analysis

1. Many more degrees of freedom than in CDCL, e.g.:
   - Choice of Boolean rule (division, saturation, or combination?)
   - Learn general PB constraints or more limited form?
   - How far to backjump when learned constraint is asserting at several levels?
   - How large precision to use in integer arithmetic?

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   - How far to backjump when learned constraint is asserting at several levels?
   - How large precision to use in integer arithmetic?
2. How to assess quality of learned constraints?
3. Theoretical potential & limitations poorly understood [VEG+18]
   - Separations in deductive power between different methods of pseudo-Boolean reasoning?
   - In particular, is division-based reasoning stronger than saturation-based reasoning? [GNY19]

Some References for Further Reading (and Watching)

Handbook of Satisfiability [BHvMW21]

- Chapter 7: Proof Complexity and SAT Solving
- Chapter 23: MaxSAT, Hard and Soft Constraints
- Chapter 24: Maximum Satisfiability
- Chapter 28: Pseudo-Boolean and Cardinality Constraints

Video tutorials on pseudo-Boolean solving

From the *Satisfiability: Theory, Practice, and Beyond* program at UC Berkeley in spring 2021
`https://tinyurl.com/PBSATtutorial`

Summing up

- Pseudo-Boolean framework expressive and powerful
- Can be approached using successful conflict-driven paradigm from SAT solving
- In theory, potential for exponential increase in performance
- In practice, some highly nontrivial challenges regarding
  - Algorithm design
  - Efficient implementation
  - Theoretical understanding
- But maybe also quite a bit of low-hanging fruit?
- And in any case lots of fun questions to work on! ☺
  (Potentially also for BSc or MSc thesis projects)

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Thank you for your attention!

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