Narrow Proofs May Be Spacious: Separating Space and Width in Resolution

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Executive Summary of Talk (1 / 2)

Resolution: proof system for refuting CNF formulas
Perhaps *the* most studied system in proof complexity
Also used in many real-world automated theorem provers

- Haken (1985): exponential lower bound on proof length (# clauses in a resolution proof)
- Ben-Sasson & Wigderson (1999): strong correlation between proof length and proof width (size of largest clause in proof)
- Results on width lead to question whether other complexity measures could yield interesting insights as well

Executive Summary of Talk (2 / 2)

- Esteban & Torán (1999): proof space
  (maximal # clauses in memory while verifying proof)
- Many lower bounds for space proven
  All turned out to match width bounds!
  Coincidence?
- Atserias & Dalmau (2003): space $\geq$ width $-$ constant for k-CNF formulas
- Problem left open: Do space and width coincide or not?

We resolve this question: separation of space and width

Outline

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \ldots \vee a_k$: set of literals
  At most k literals: k-clause
- CNF formula $F = C_1 \wedge \ldots \wedge C_m$: set of clauses
  k-CNF formula: CNF formula consisting of k-clauses
  (assume k fixed)
- $F \vDash D$: semantical implication, $\alpha(F)$ true $\Rightarrow \alpha(D)$ true for all truth value assignments α
- $[n] = \{1, 2, \ldots, n\}$

Resolution Derivation

Sequence of sets of clauses, or clause configurations, $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for clause $C \in F$ (axiom)

Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for clause $C \in \mathbb{C}_{t-1}$

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$$\frac{C \vee x \quad D \vee \overline{x}}{C \vee D}$$

Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F, i.e., $0 \in \mathbb{C}_\tau$

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Length, Width and Space

- Length $L(\pi)$ of resolution refutation $\pi : F \vdash 0$
  # distinct clauses in all of π
- Width $W(\pi)$ of resolution refutation $\pi : F \vdash 0$
  # literals in largest clause in π
- Space $Sp(\pi)$ of resolution refutation $\pi : F \vdash 0$
  # clauses in largest clause configuration $\mathbb{C}_t \in \pi$

Length, Width and Space of Refuting F

- Length of refuting F is

$$L(F \vdash 0) = \min_{\pi : F \vdash 0} \{L(\pi)\}$$

- Width of refuting F is

$$W(F \vdash 0) = \min_{\pi : F \vdash 0} \{W(\pi)\}$$

- Space of refuting F is

$$Sp(F \vdash 0) = \min_{\pi : F \vdash 0} \{Sp(\pi)\}$$

$$\begin{aligned} L(F \vdash 0) &\leq 2^{(\#\text{ variables in } F + 1)} \\ W(F \vdash 0) &\leq \#\text{ variables in } F \\ Sp(F \vdash 0) &\leq \min(\#\text{ variables in } F, \#\text{ clauses in } F) + \mathcal{O}(1) \end{aligned}$$

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Connection between Length and Width

A resolution proof in small width is necessarily short.
For a proof in width w, $\left(2 \cdot |Vars(F)|\right)^w$ is an upper bound on the number of possible clauses.

There is a kind of converse to this:

Theorem (Ben-Sasson & Wigderson 1999)

The width of refuting a k-CNF formula F over n variables is

$$W(F \vdash 0) = \mathcal{O}\left(\sqrt{n \log L(F \vdash 0)}\right).$$

This bound on width in terms of length is essentially optimal (Bonet & Galesi 1999).

Connection between Space and Width

All previously shown lower bounds on space coincide with lower bounds on width—true in general?

Theorem (Atserias & Dalmau 2003)

For any unsatisfiable k-CNF formula F it holds that

$$Sp(F \vdash 0) \geq W(F \vdash 0) - \mathcal{O}(1).$$

But do space and width always coincide?
Or is there a k-CNF formula family $\{F_n\}_{n=1}^{\infty}$ such that $Sp(F_n \vdash 0) = \omega(W(F_n \vdash 0))$?

Pebbles Games on Graphs

One-player game played on directed acyclic graphs (DAGs)

- Devised for studying programming languages and compiler construction
- Have found a variety of applications in complexity theory

Conventions

- $V(G)$ denote the vertices of a DAG G
- vertices with indegree 0 are sources
- vertices with outdegree 0 are targets

This talk: Only consider DAGs with single target z and all non-source vertices having indegree 2

Formal Definition of Pebble Game

Pebble configuration: pair of subsets $\mathbb{P} = (B, W)$ of black- and white-pebbled vertices

Black-white pebbling: sequence $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ such that $\mathbb{P}_0 = (\emptyset, \emptyset)$ and $\mathbb{P}_t$ follows from $\mathbb{P}_{t-1}$ by one of the rules:

1. If all immediate predecessors of an empty vertex v have pebbles on them, a black pebble can be placed on v.
2. A black pebble can always be removed from any vertex.
3. A white pebble can always be placed on any empty vertex.
4. If all immediate predecessors of a white-pebbled vertex v are pebbled, the white pebble on v can be removed.

Goal: reach $\mathbb{P}_\tau = (\{z\}, \emptyset)$ using few pebbles

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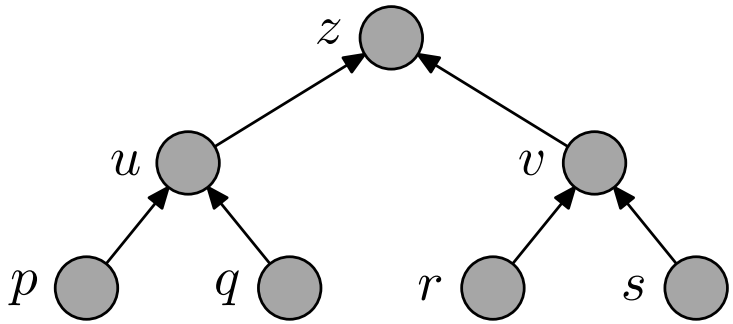
Example Pebbling and Pebbling Price



- Cost of pebbling $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$: max # pebbles in any $\mathbb{P}_t = (B_t, W_t)$
- Black-white pebbling price $\mathit{BW\text{-}Peb}(G)$ of DAG G is minimal cost of any pebbling reaching $(\{z\}, \emptyset)$
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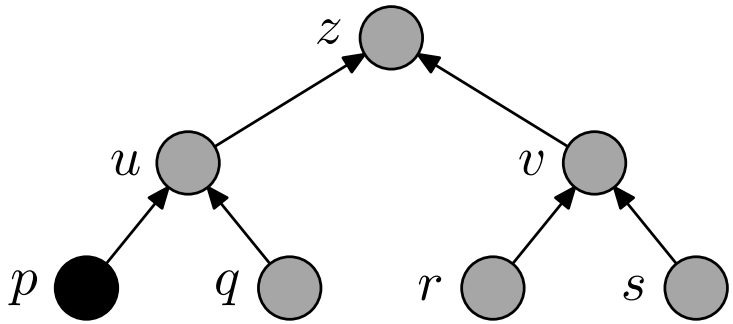

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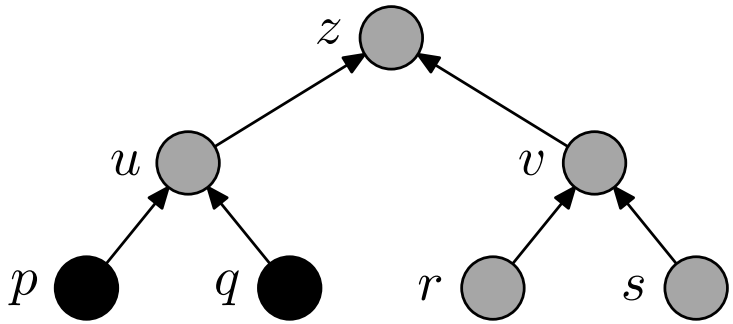

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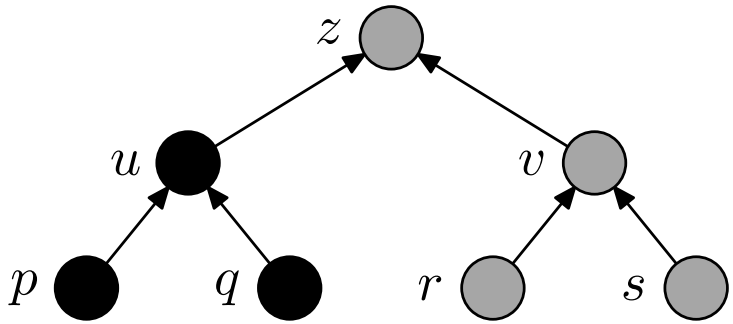

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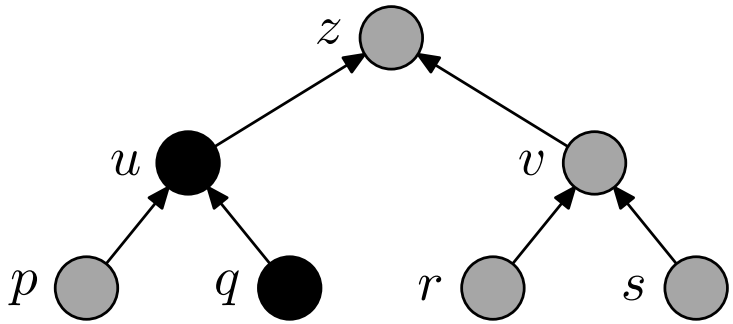

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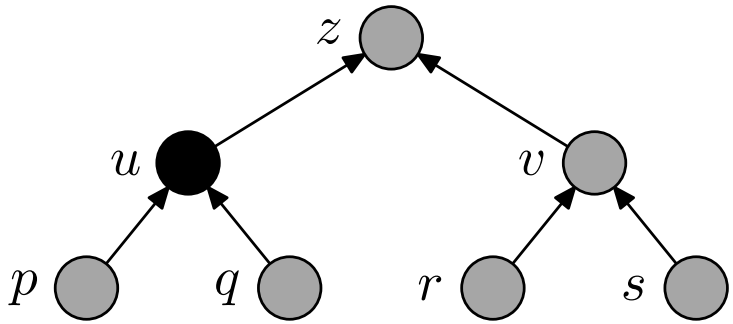

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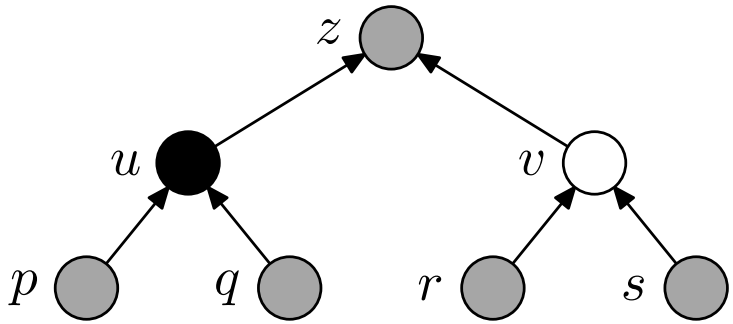

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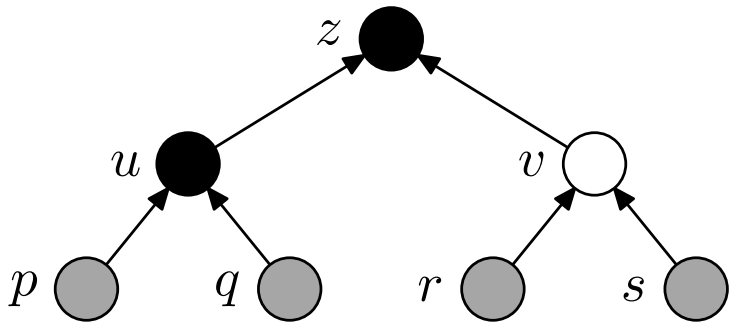

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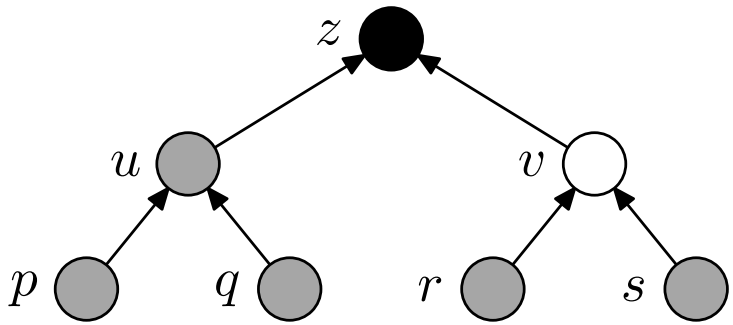

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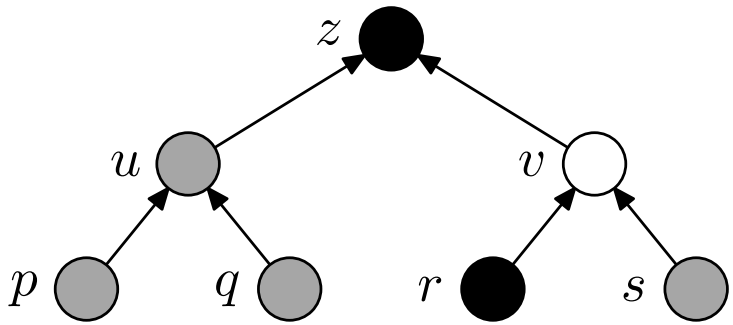

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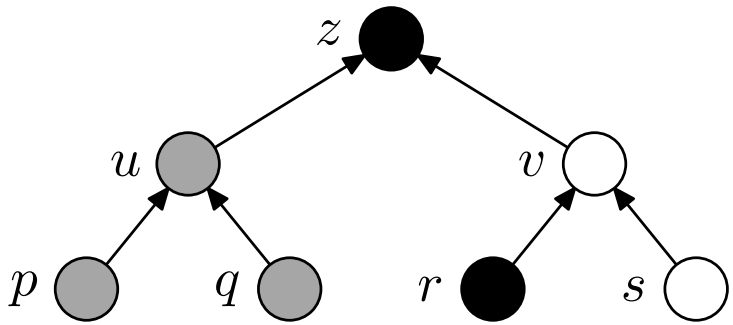

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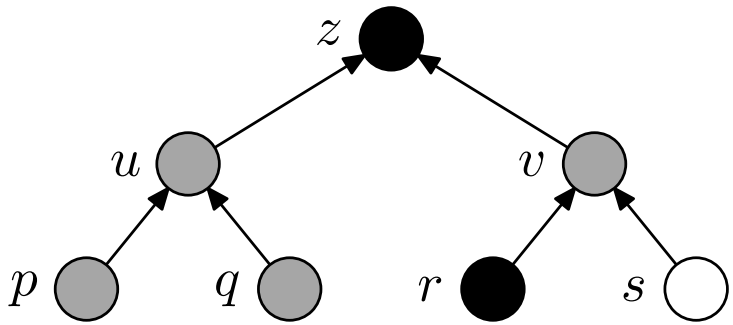

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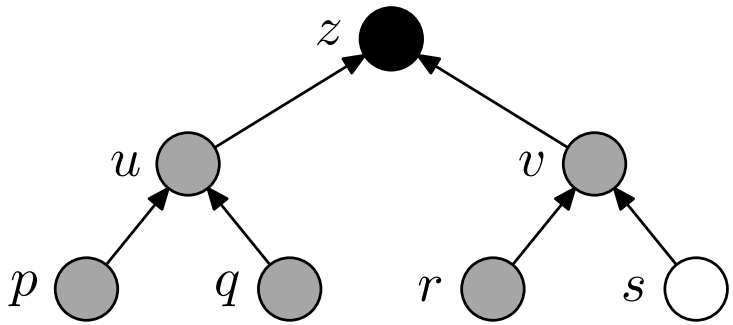

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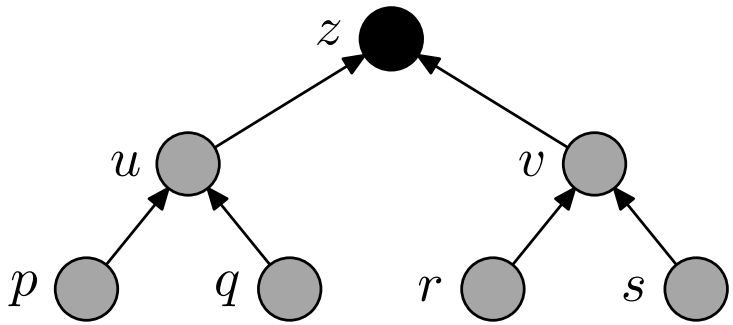

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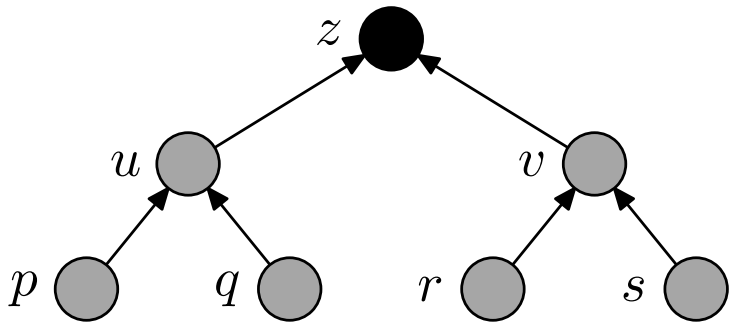

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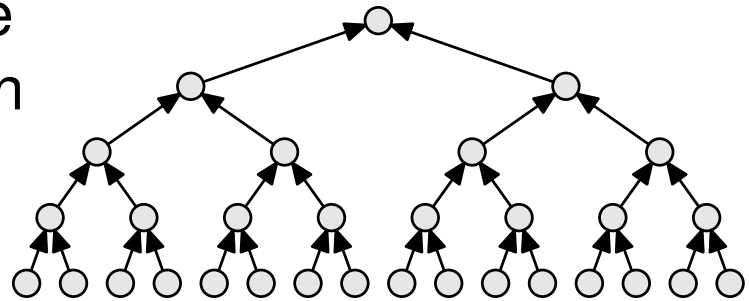
Pebbling Price of Binary Trees

Let T_h denote complete binary tree of height h considered as DAG with edges directed towards root



- Pebbling price of T_h is

$$Peb(T_h) = h + 2$$

- Black-white pebbling price of T_h is

$$BW\text{-}Peb(T_h) = \left\lfloor \frac{h}{2} \right\rfloor + 3 = \Omega(h)$$

(Lengauer & Tarjan 1980)

Definition of Pebbling Contradiction

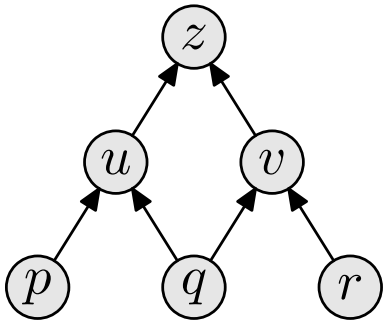
Pebbling contradiction: CNF formula encoding pebble game on DAG G with sources S, unique target z and all non-source vertices having indegree 2

Associate d variables $v_1, \ldots, v_d$ with every vertex $v \in V(G)$

The dth degree pebbling contradiction Peb_G^d over G is the conjunction of the following clauses:

- $\bigvee_{i=1}^{d} s_i$ for all $s \in S$ (source axioms)
- $\overline{u}_i \vee \overline{v}_j \vee \bigvee_{l=1}^{d} w_l$ for all $i, j \in [d]$ and all $w \in V(G) \setminus S$, where u, v are the two predecessors of w (pebbling axioms)
- $\overline{z}_i$ for all $i \in [d]$ (target axioms)

Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



$$
\begin{aligned}
&(p_1 \vee p_2) \\
\wedge\ &(q_1 \vee q_2) \\
\wedge\ &(r_1 \vee r_2) \\
\wedge\ &(\overline{p}_1 \vee \overline{q}_1 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{p}_1 \vee \overline{q}_2 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{p}_2 \vee \overline{q}_1 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{p}_2 \vee \overline{q}_2 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{q}_1 \vee \overline{r}_1 \vee v_1 \vee v_2) \\
\wedge\ &(\overline{q}_1 \vee \overline{r}_2 \vee v_1 \vee v_2)
\end{aligned}
$$

$$
\begin{aligned}
\wedge\ &(\overline{q}_2 \vee \overline{r}_1 \vee v_1 \vee v_2) \\
\wedge\ &(\overline{q}_2 \vee \overline{r}_2 \vee v_1 \vee v_2) \\
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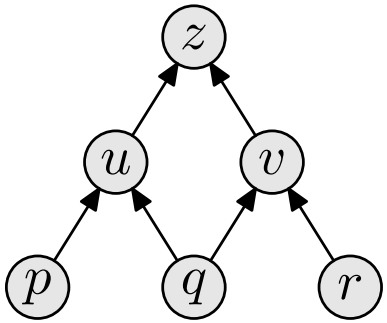
\wedge\ &(\overline{u}_2 \vee \overline{v}_2 \vee z_1 \vee z_2) \\
\wedge\ &\overline{z}_1 \\
\wedge\ &\overline{z}_2
\end{aligned}
$$

Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



$$
\begin{aligned}
&(p_1 \vee p_2) \\
\wedge\ &(q_1 \vee q_2) \\
\wedge\ &(r_1 \vee r_2) \\
\wedge\ &(\overline{p}_1 \vee \overline{q}_1 \vee u_1 \vee u_2) \\
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\wedge\ &(\overline{p}_2 \vee \overline{q}_2 \vee u_1 \vee u_2) \\
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$$

$$
\begin{aligned}
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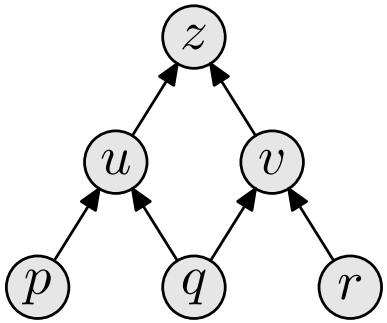
\wedge\ &\overline{z}_1 \\
\wedge\ &\overline{z}_2
\end{aligned}
$$

Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



$$
\begin{aligned}
&(p_1 \vee p_2) \\
\wedge\ &(q_1 \vee q_2) \\
\wedge\ &(r_1 \vee r_2) \\
\wedge\ &(\overline{p}_1 \vee \overline{q}_1 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{p}_1 \vee \overline{q}_2 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{p}_2 \vee \overline{q}_1 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{p}_2 \vee \overline{q}_2 \vee u_1 \vee u_2) \\
\wedge\ &(\overline{q}_1 \vee \overline{r}_1 \vee v_1 \vee v_2) \\
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$$

$$
\begin{aligned}
\wedge\ &(\overline{q}_2 \vee \overline{r}_1 \vee v_1 \vee v_2) \\
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\wedge\ &(\overline{u}_1 \vee \overline{v}_1 \vee z_1 \vee z_2) \\
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\wedge\ &(\overline{u}_2 \vee \overline{v}_2 \vee z_1 \vee z_2) \\
\wedge\ &\overline{z}_1 \\
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\end{aligned}
$$

Pebbling Contradictions Easy w.r.t. Length and Width

Peb_G^d is an unsatisfiable (2+d)-CNF formula with

- $d \cdot |V(G)|$ variables
- $\mathcal{O}\left(d^2 \cdot |V(G)|\right)$ clauses

Can be refuted by deriving $\bigvee_{i=1}^{d} v_i$ for all $v \in V(G)$ inductively in topological order and resolving with target axioms $\overline{z}_i$, $i \in [d]$

It follows that

- $L(F \vdash 0) = \mathcal{O}\left(d^2 \cdot |V(G)|\right)$
- $W(F \vdash 0) = \mathcal{O}(d)$

(Ben-Sasson et al. 2000)

What about Pebbling Contradictions and Space?

Upper bounds:

- Arbitrary DAGs G
  optimal black pebbling of G + proof from previous slide:
  $Sp\left(Peb_G^d \vdash 0\right) \leq Peb(G) + \mathcal{O}(1)$
- Binary trees T_h
  improvement by Esteban & Torán (2003):
  $Sp\left(Peb_{T_h}^2 \vdash 0\right) \leq \left\lceil \frac{2h+1}{3} \right\rceil + 3 = \frac{2}{3} Peb(T_h) + \mathcal{O}(1)$
- Only one variable / vertex
  Ben-Sasson (2002):
  $Sp\left(Peb_G^1 \vdash 0\right) = \mathcal{O}(1)$ for arbitrary G

No lower bounds on space for $d \geq 2$ previously known

Our Results

Theorem

Let $Peb_{T_h}^d$ *denote the pebbling contradiction of degree* $d \geq 2$ *defined over the complete binary tree of height* h*. Then the space of refuting* $Peb_{T_h}^d$ *in resolution is* $Sp\left(Peb_{T_h}^d \vdash 0\right) = \Theta(h)$.

Corollary

For all $k \geq 4$*, there is a family of* k*-CNF formulas* $\left\{F_n\right\}_{n=1}^{\infty}$ *of size* $\mathcal{O}(n)$ *with refutation width* $W(F_n \vdash 0) = \mathcal{O}(1)$ *and refutation space* $Sp(F_n \vdash 0) = \Theta(\log n)$.

Proof Idea

Prove lower bounds on space of $\pi : Peb_G^d \vdash 0$ by

1. Interpreting clause configurations $\mathbb{C}_t \in \pi$ in terms of black and white pebbles on G
2. Showing that if $\mathbb{C}_t$ induces N black and white pebbles it contains at least N clauses (if $d \geq 2$)
3. Establishing that $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ induces black-white pebbling $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ (works only for binary trees T_h)

Then some $\mathbb{C}_t \in \pi$ must induce $BW\text{-}Peb(T_h)$ pebbles

$$\Downarrow$$

$$|\mathbb{C}_t| \geq BW\text{-}Peb(T_h) = \Omega(h)$$

$$\Downarrow$$

$$Sp\left(Peb_{T_h}^d \vdash 0\right) = \Omega(h)$$

Proof Idea

Prove lower bounds on space of $\pi : \mathit{Peb}_G^d \vdash 0$ by

1. Interpreting clause configurations $\mathbb{C}_t \in \pi$ in terms of black and white pebbles on G
2. Showing that if $\mathbb{C}_t$ induces N black and white pebbles it contains at least N clauses (if $d \geq 2$)
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Then some $\mathbb{C}_t \in \pi$ must induce $\mathit{BW\text{-}Peb}(T_h)$ pebbles

$$\Downarrow$$

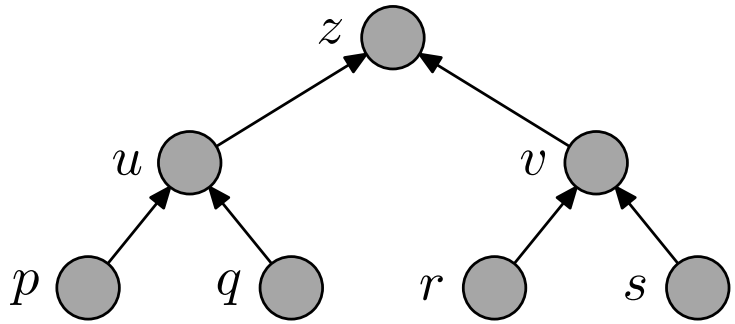
$$|\mathbb{C}_t| \geq \mathit{BW\text{-}Peb}(T_h) = \Omega(h)$$

$$\Downarrow$$

$$\mathit{Sp}\left(\mathit{Peb}_{T_h}^d \vdash 0\right) = \Omega(h)$$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target



$$\left[\quad\right]$$

Empty start configuration

Developing an Intuition for Black Pebbles

1.	p_1	Source
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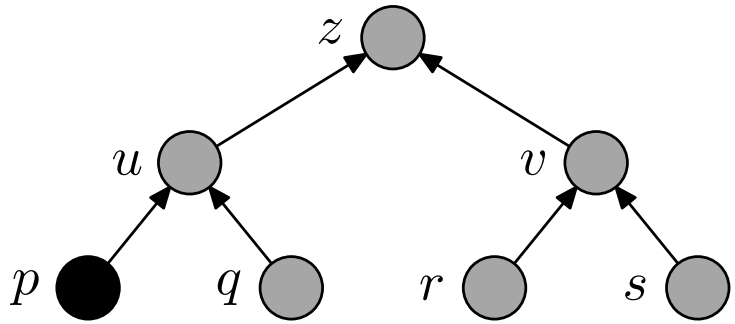

$$\left[\begin{array}{l} p_1 \\ \\ \\ \\ \end{array}\right]$$

Download axiom 1: p_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
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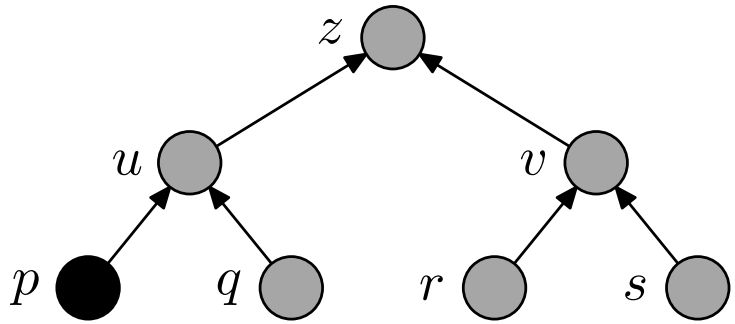

$$\begin{bmatrix} p_1 \\ \\ \\ \\ \end{bmatrix}$$

Download axiom 1: p_1

Developing an Intuition for Black Pebbles

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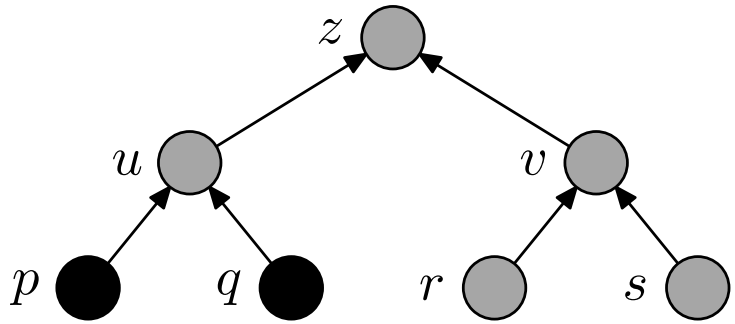

$$\begin{bmatrix} p_1 \\ q_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 2: q_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
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5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
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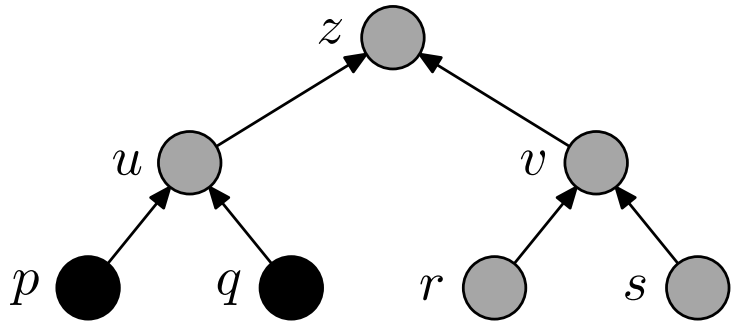

$$\begin{bmatrix} p_1 \\ q_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 2: q_1

Developing an Intuition for Black Pebbles

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2.	q_1	Source
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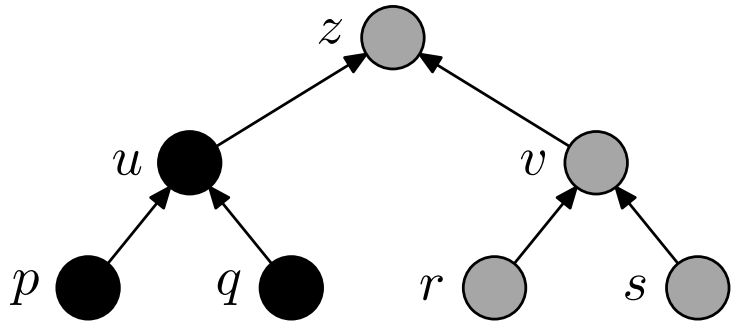

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Download axiom 5: $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

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2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
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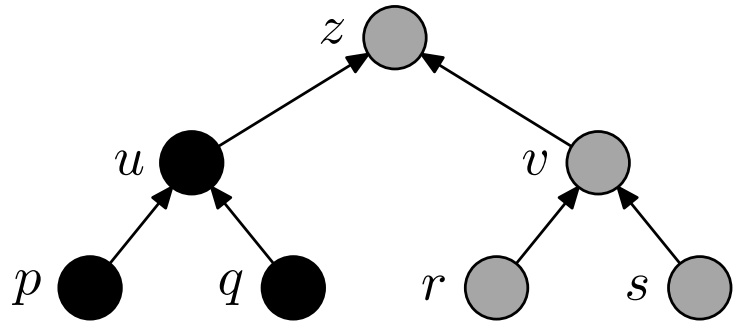

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Download axiom 5: $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

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2.	q_1	Source
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5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
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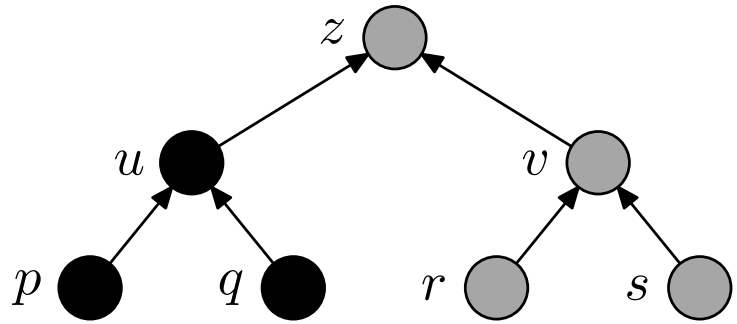
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target



$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Infer $\overline{q}_1 \vee u_1$ from
p_1 and $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
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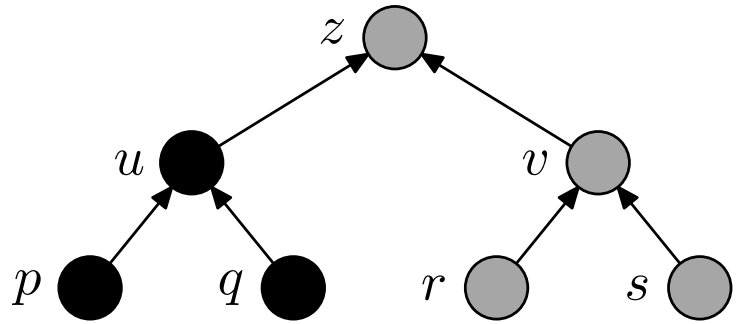

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{q}_1 \vee u_1 \end{bmatrix}$$

Infer $\overline{q}_1 \vee u_1$ from
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Developing an Intuition for Black Pebbles

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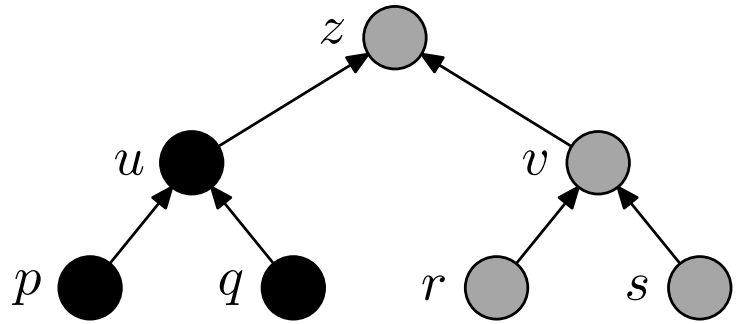

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{q}_1 \vee u_1 \end{bmatrix}$$

Infer $\overline{q}_1 \vee u_1$ from
p_1 and $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
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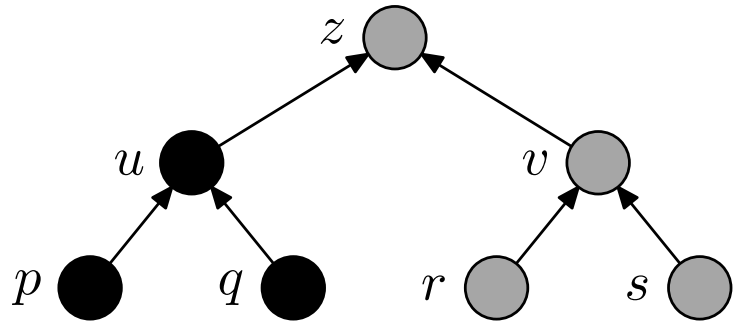

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{q}_1 \vee u_1 \end{bmatrix}$$

Erase clause p_1

Developing an Intuition for Black Pebbles

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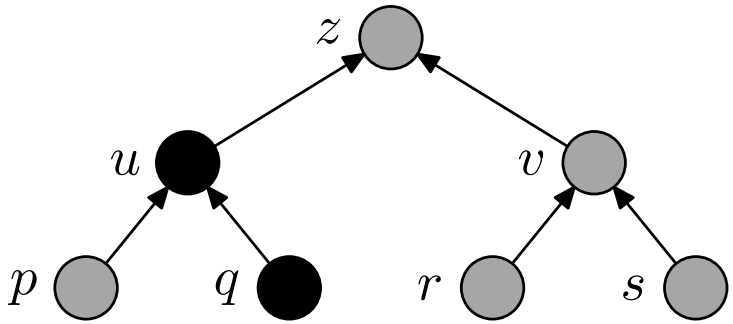

$$\begin{bmatrix} q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Erase clause p_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
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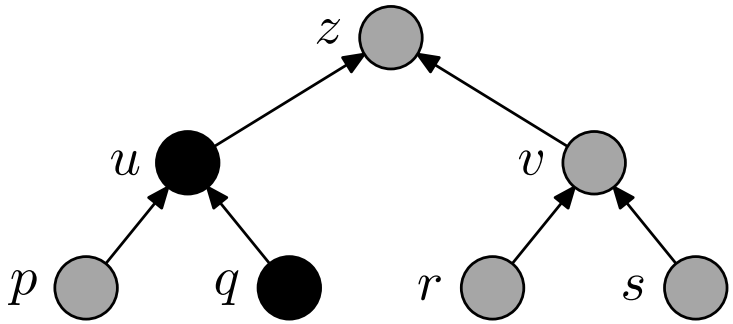

$$\begin{bmatrix} q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{q}_1 \vee u_1 \end{bmatrix}$$

Erase clause $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
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6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
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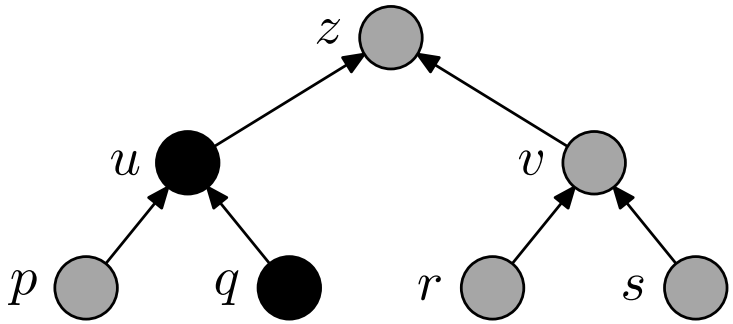
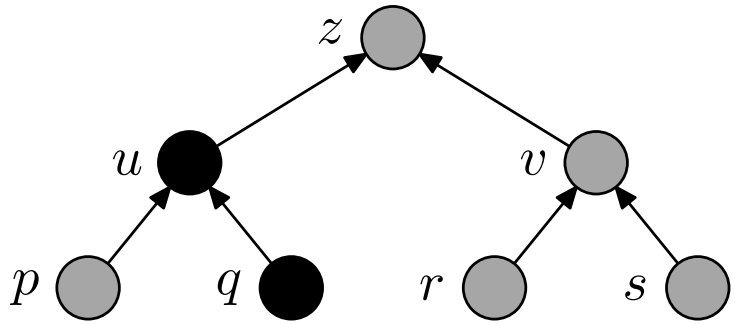


$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \end{bmatrix}$$

Erase clause $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

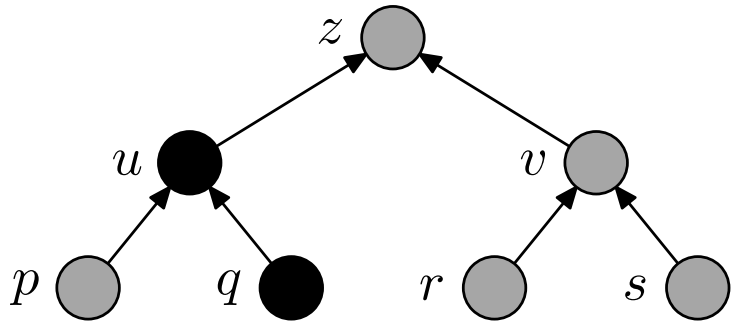


$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \end{bmatrix}$$

Infer u_1 from
q_1 and $\overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

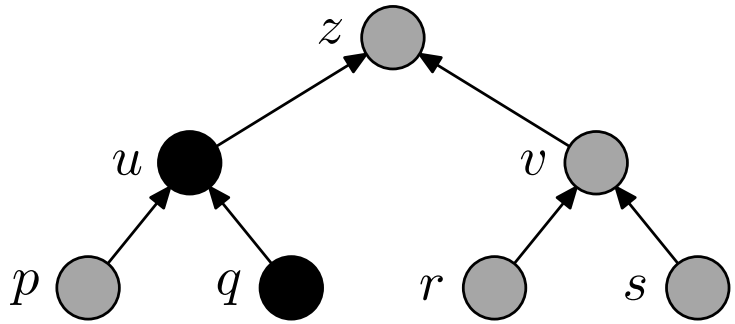


$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \\ u_1 \\ \\ \end{bmatrix}$$

Infer u_1 from
q_1 and $\overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

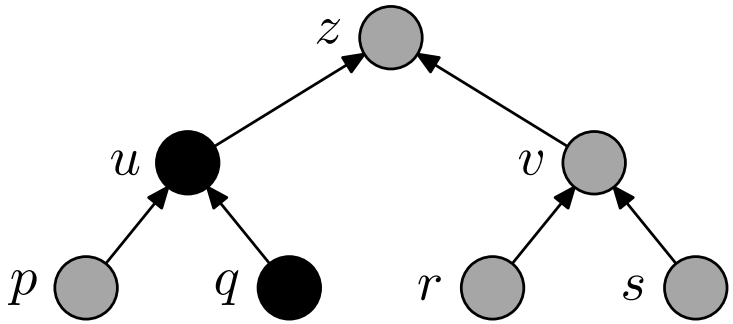


$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \\ u_1 \\ \\ \end{bmatrix}$$

Infer u_1 from
q_1 and $\overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

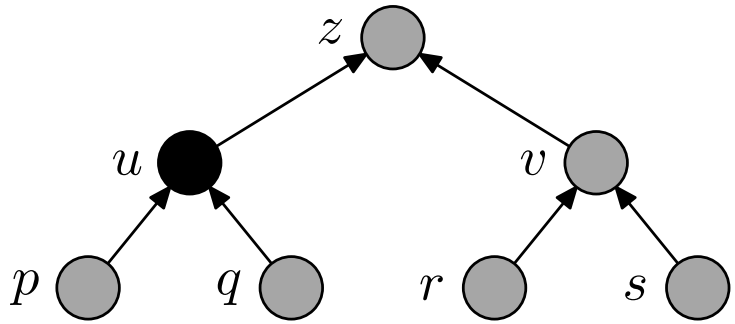


$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \\ u_1 \\ \\ \end{bmatrix}$$

Erase clause q_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

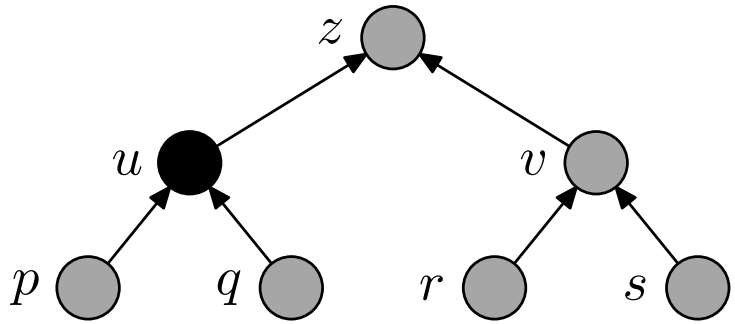


$$\begin{bmatrix} \overline{q}_1 \vee u_1 \\ u_1 \\ \\ \\ \\ \end{bmatrix}$$

Erase clause q_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

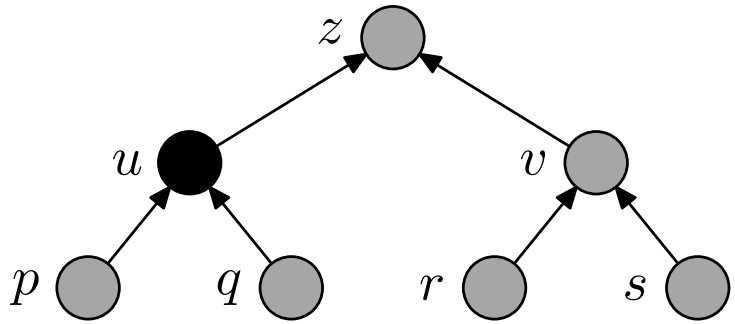


$$\begin{bmatrix} \overline{q}_1 \vee u_1 \\ u_1 \end{bmatrix}$$

Erase clause $\overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

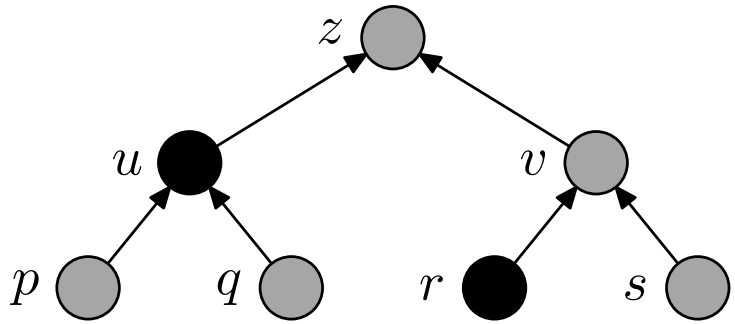


$$\begin{bmatrix} u_1 \\ \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

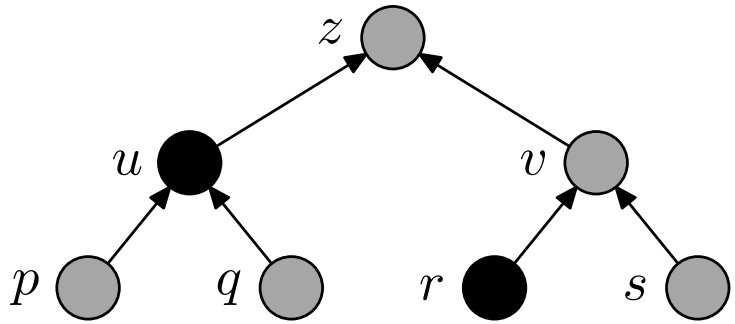


$$\begin{bmatrix} u_1 \\ r_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 3: r_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

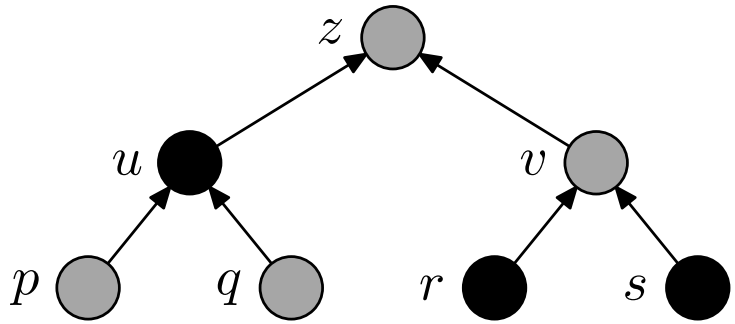


$$\begin{bmatrix} u_1 \\ r_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 3: r_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

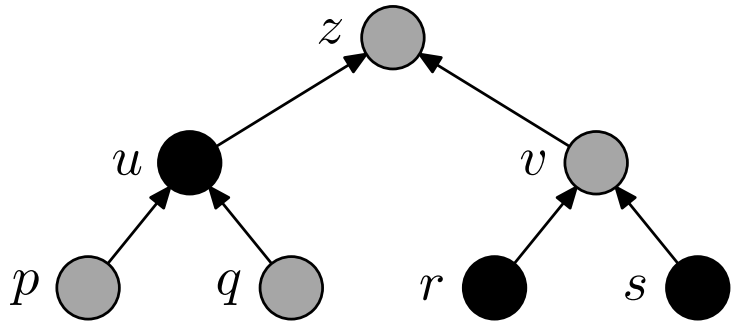


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 4: s_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

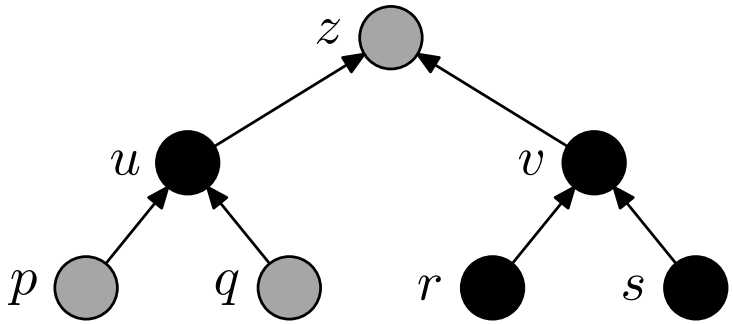


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \\ \end{bmatrix}$$

Download axiom 4: s_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

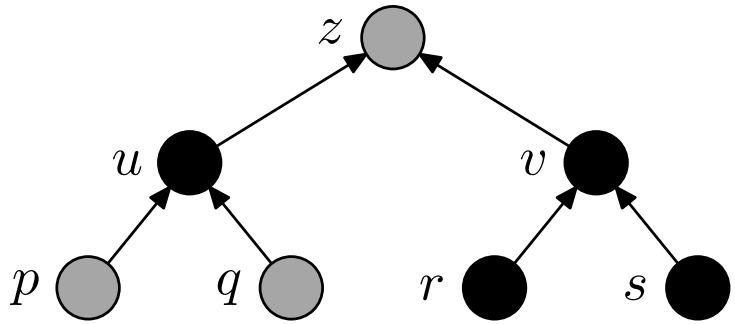


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \end{bmatrix}$$

Download axiom 6: $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

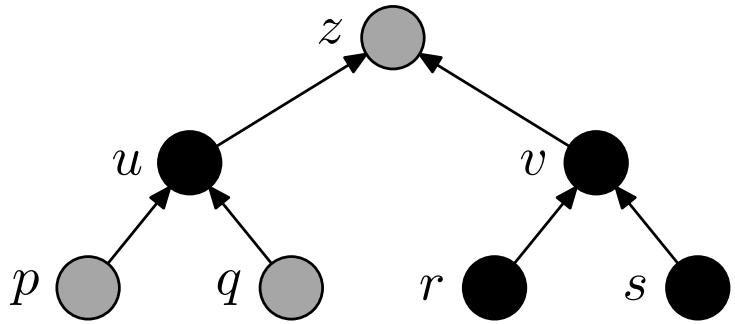


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \end{bmatrix}$$

Download axiom 6: $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

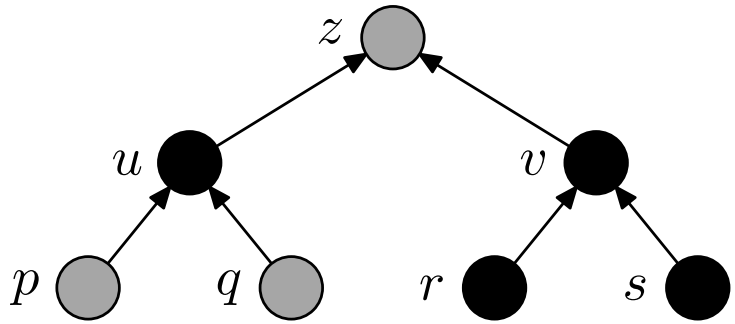


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \end{bmatrix}$$

Infer $\overline{s}_1 \vee v_1$ from
r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

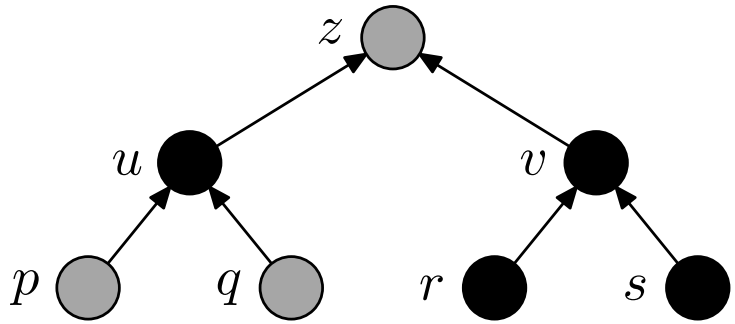


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{s}_1 \vee v_1 \end{bmatrix}$$

Infer $\overline{s}_1 \vee v_1$ from
r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

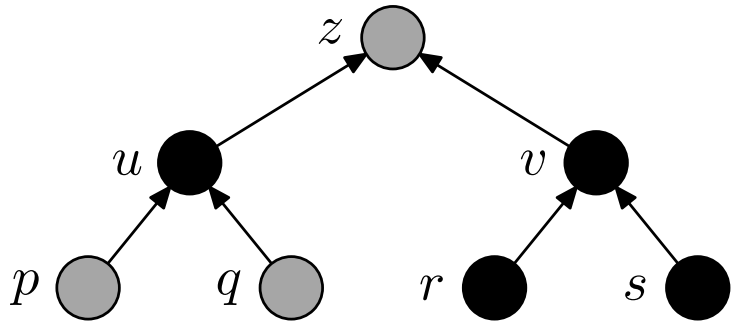


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{s}_1 \vee v_1 \end{bmatrix}$$

Infer $\overline{s}_1 \vee v_1$ from
r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

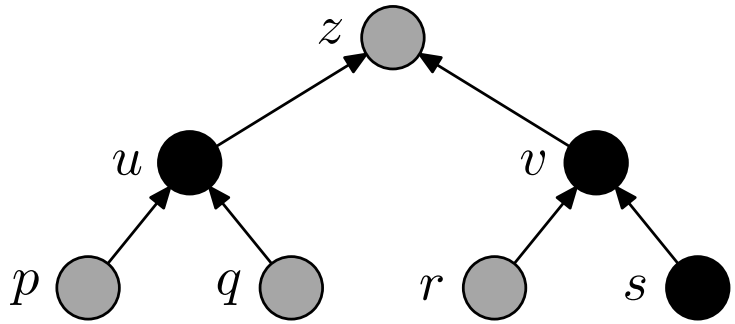


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{s}_1 \vee v_1 \end{bmatrix}$$

Erase clause r_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

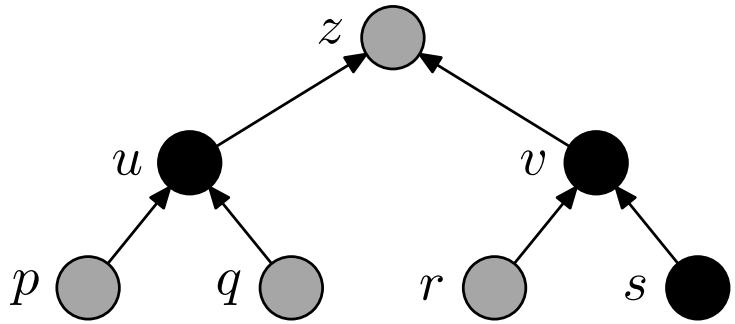


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{s}_1 \vee v_1 \\ \end{bmatrix}$$

Erase clause r_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

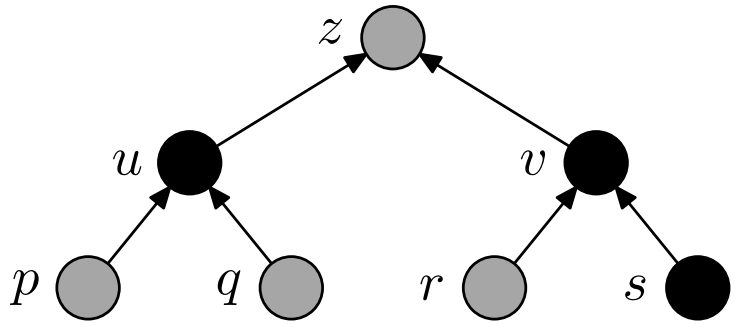


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{s}_1 \vee v_1 \\ \end{bmatrix}$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

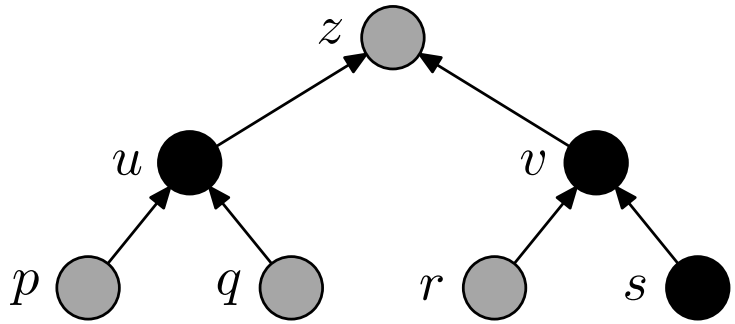


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \vee v_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

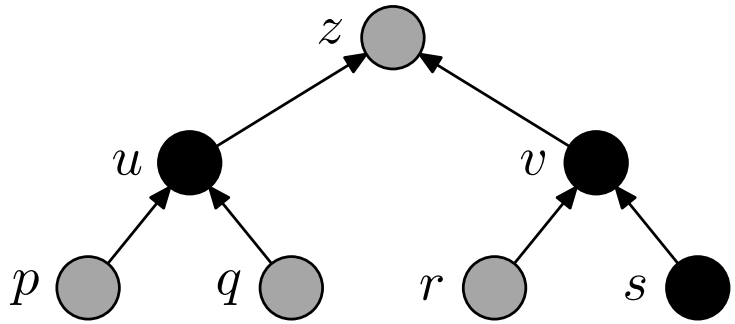


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \vee v_1 \end{bmatrix}$$

Infer v_1 from
s_1 and $\overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

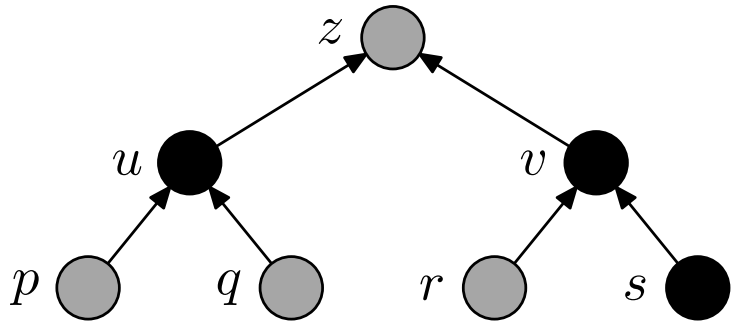


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \vee v_1 \\ v_1 \\ \\ \end{bmatrix}$$

Infer v_1 from
s_1 and $\overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

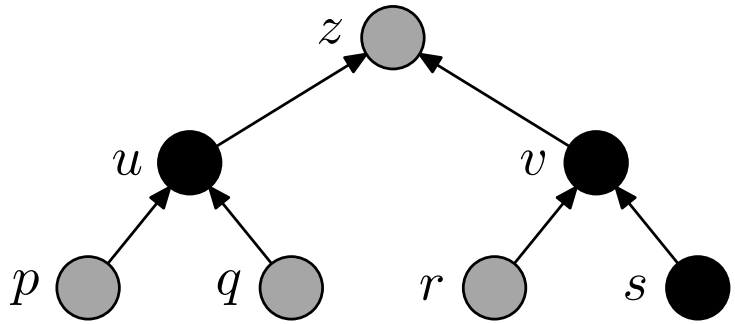


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \vee v_1 \\ v_1 \\ \\ \end{bmatrix}$$

Infer v_1 from
s_1 and $\overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

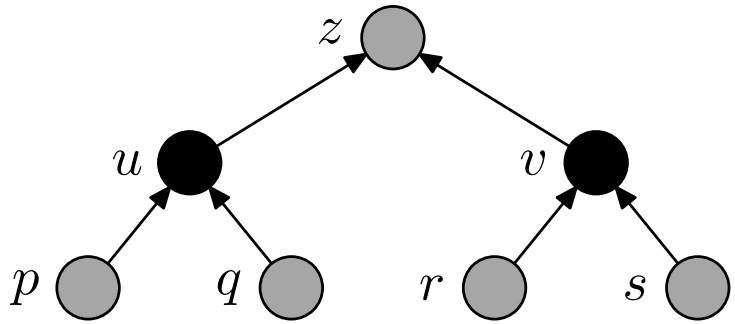


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \vee v_1 \\ v_1 \\ \\ \end{bmatrix}$$

Erase clause s_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

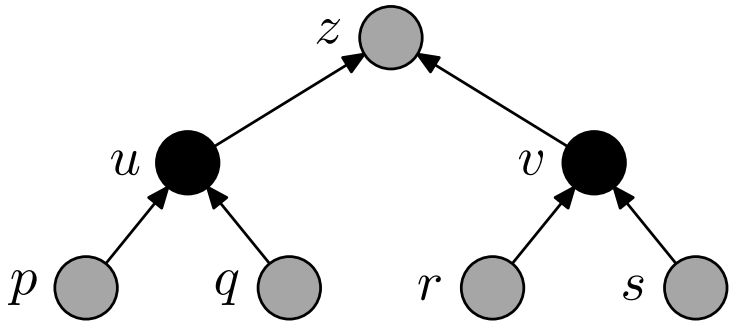


$$\begin{bmatrix} u_1 \\ \overline{s}_1 \vee v_1 \\ v_1 \\ \\ \end{bmatrix}$$

Erase clause s_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

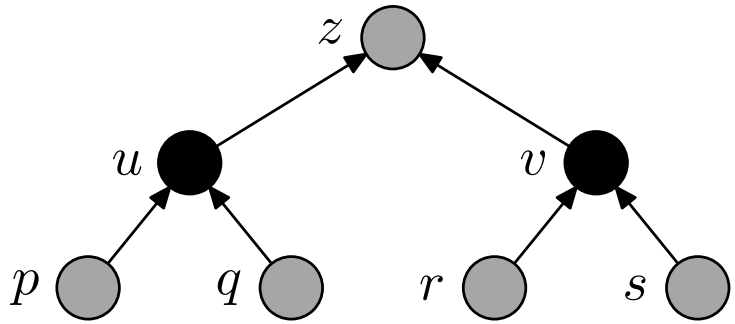


$$\begin{bmatrix} u_1 \\ \overline{s}_1 \vee v_1 \\ v_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

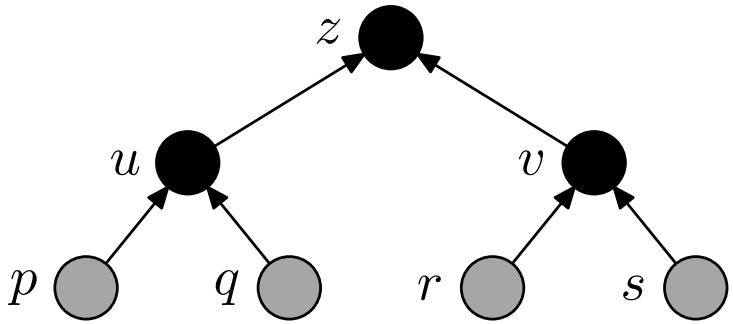


$$\begin{bmatrix} u_1 \\ v_1 \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

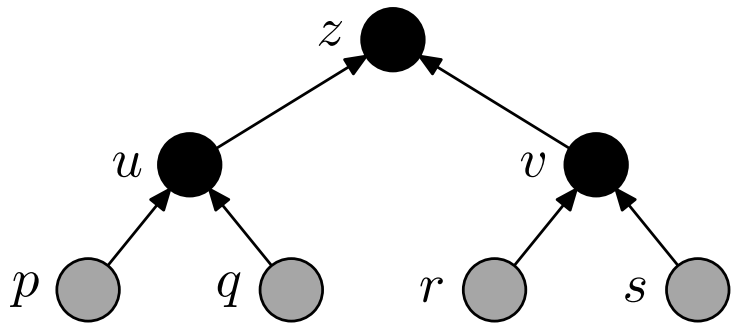


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \end{bmatrix}$$

Download axiom 7: $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

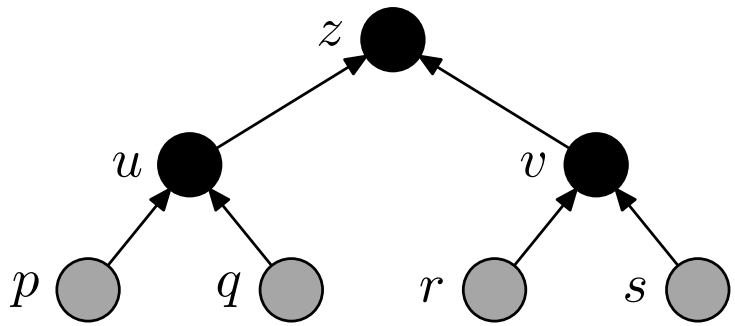


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Download axiom 7: $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

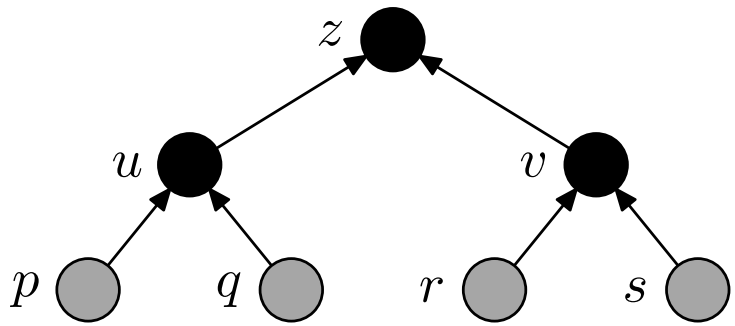


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Infer $\overline{v}_1 \vee z_1$ from
u_1 and $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

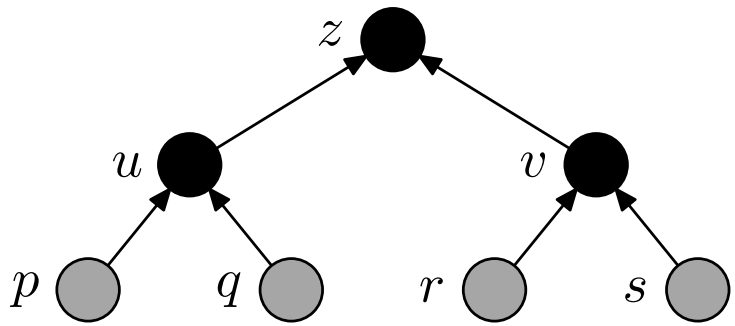


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{v}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{v}_1 \vee z_1$ from
u_1 and $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

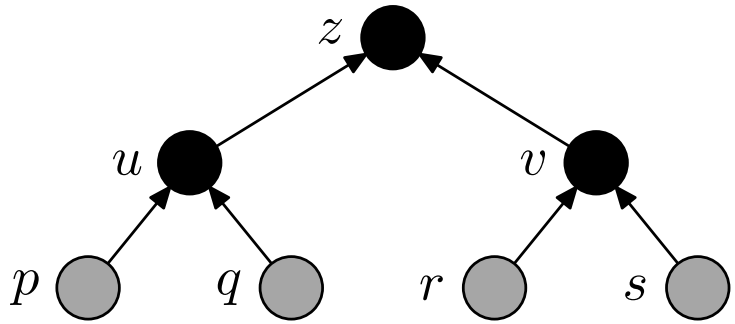


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Infer $\overline{v}_1 \vee z_1$ from
u_1 and $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

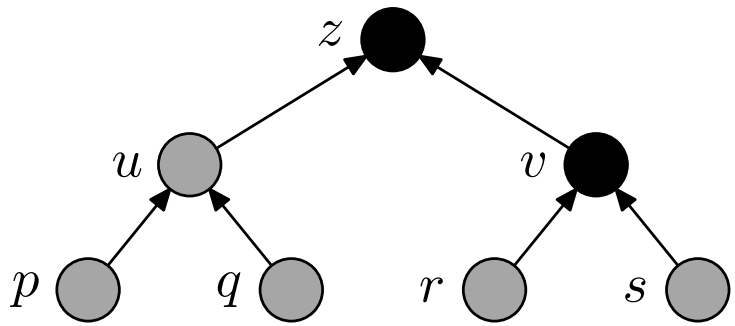


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Erase clause u_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

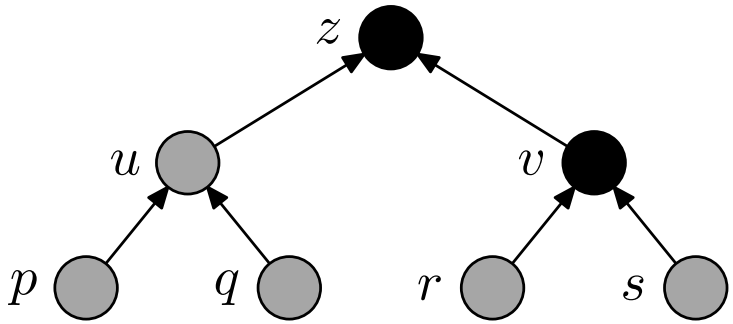


$$\begin{bmatrix} v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Erase clause u_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

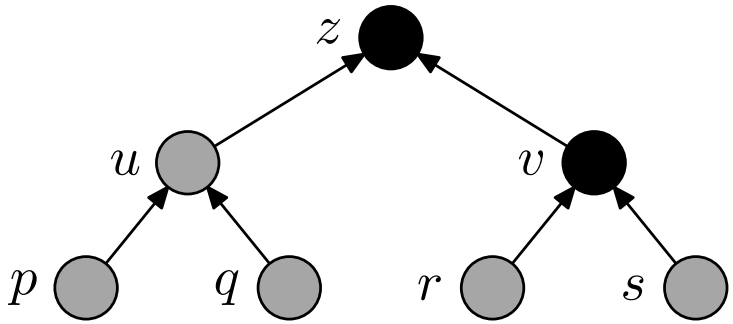


$$\begin{bmatrix} v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

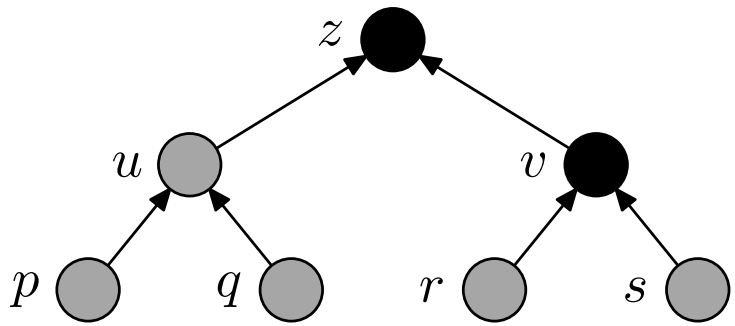


$$\begin{bmatrix} v_1 \\ \overline{v}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

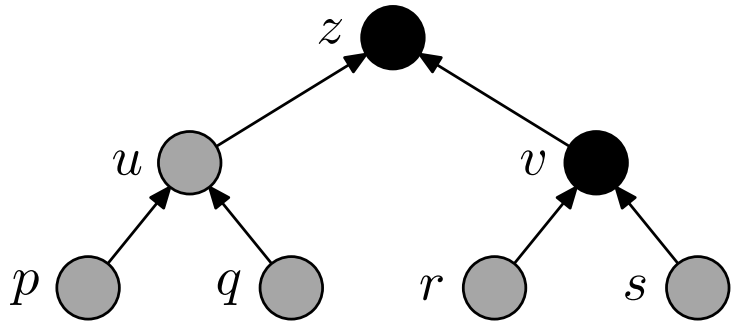


$$\begin{bmatrix} v_1 \\ \overline{v}_1 \vee z_1 \end{bmatrix}$$

Infer z_1 from
v_1 and $\overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

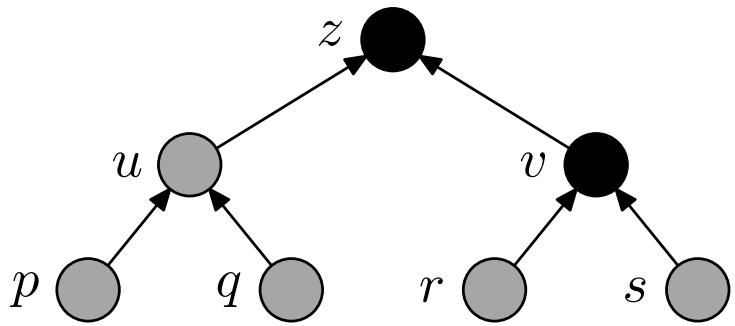


$$\begin{bmatrix} v_1 \\ \overline{v}_1 \vee z_1 \\ z_1 \end{bmatrix}$$

Infer z_1 from
v_1 and $\overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

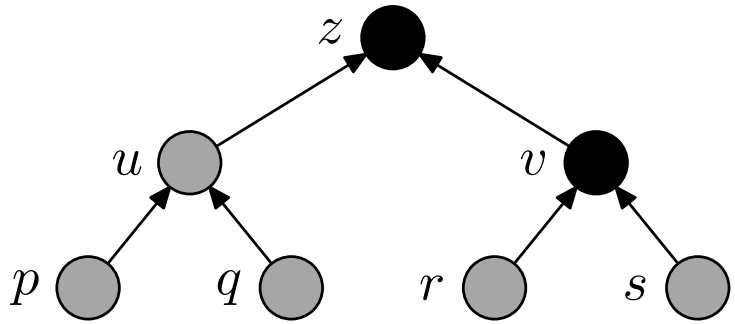


$$\begin{bmatrix} v_1 \\ \overline{v}_1 \vee z_1 \\ z_1 \end{bmatrix}$$

Infer z_1 from
v_1 and $\overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

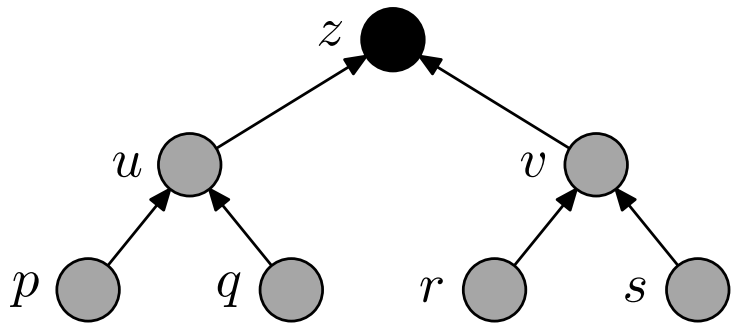


$$\begin{bmatrix} v_1 \\ \overline{v}_1 \vee z_1 \\ z_1 \\ \\ \\ \end{bmatrix}$$

Erase clause v_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

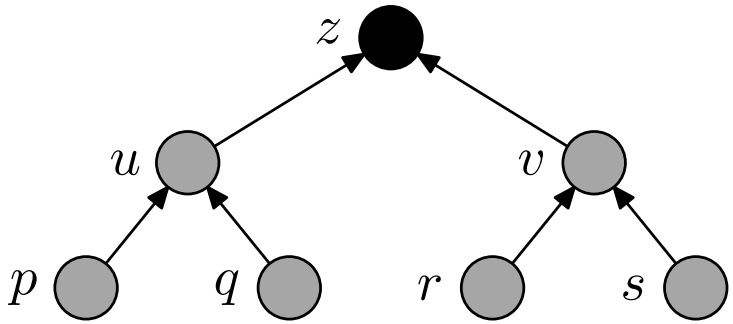


$$\begin{bmatrix} \overline{v}_1 \vee z_1 \\ z_1 \\ \\ \\ \end{bmatrix}$$

Erase clause v_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

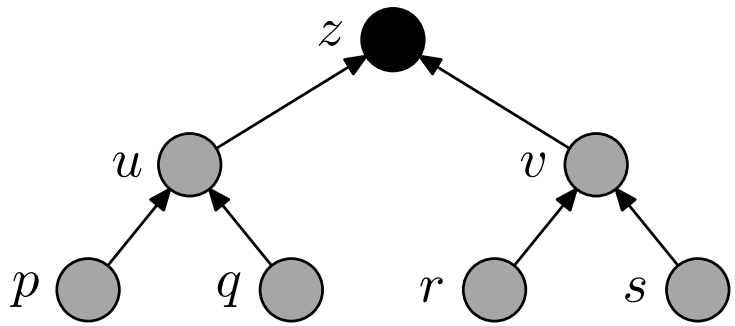


$$\begin{bmatrix} \overline{v}_1 \vee z_1 \\ z_1 \end{bmatrix}$$

Erase clause $\overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

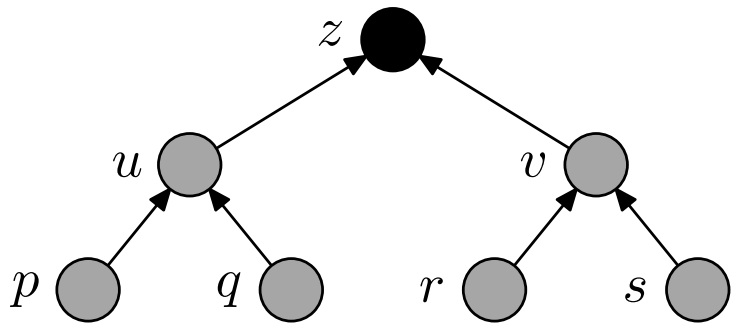


$$\begin{bmatrix} z_1 \\ \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

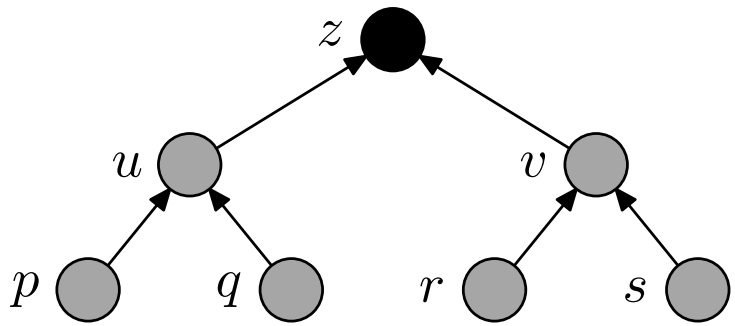


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \\ \\ \end{bmatrix}$$

Download axiom 8: $\overline{z}_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

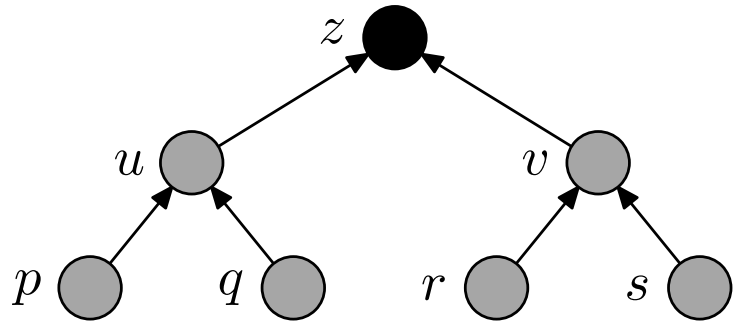


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \\ \\ \end{bmatrix}$$

Download axiom 8: $\overline{z}_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

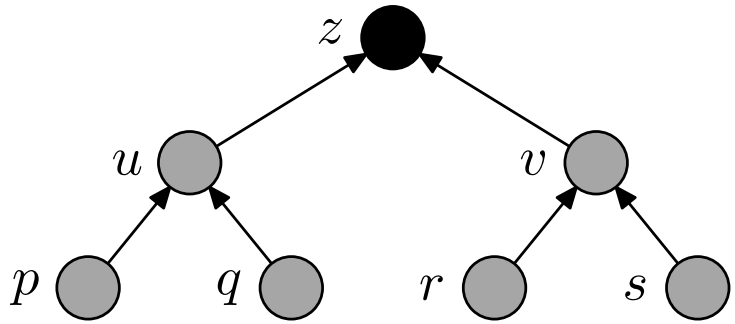


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \\ \\ \end{bmatrix}$$

Infer 0 from
z_1 and $\overline{z}_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

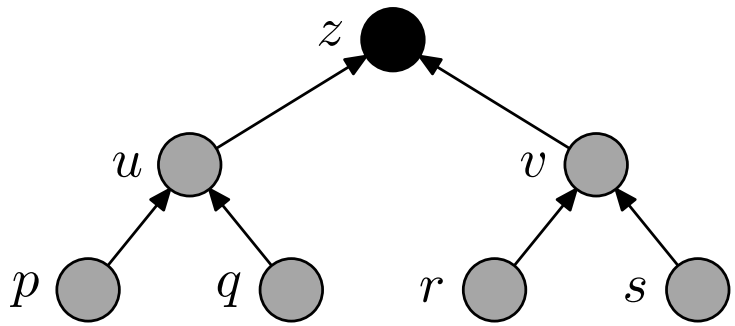


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ 0 \end{bmatrix}$$

Infer 0 from z_1 and $\overline{z}_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target



$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ 0 \\ \\ \end{bmatrix}$$

Infer 0 from z_1 and $\overline{z}_1$

Informal Description of Induced Pebbles

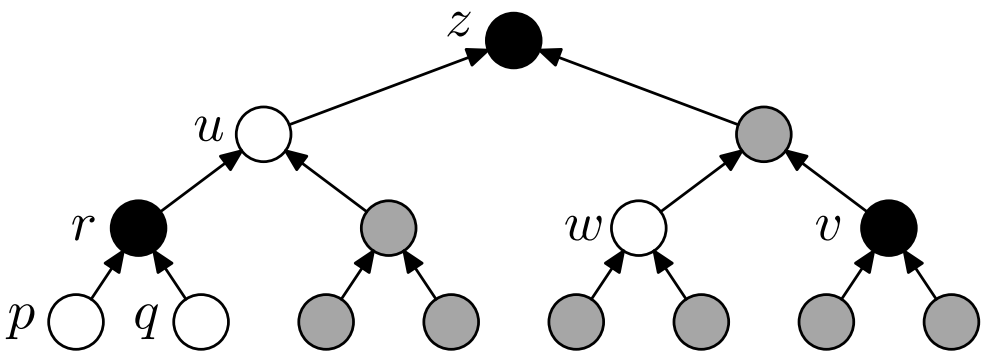
Intuition

$\mathbb{C}_t \vDash \bigvee_{i=1}^{d} v_i \Leftrightarrow$ Black pebble on v with no white pebbles below

Black pebble on v with white pebbles on W below $\Leftrightarrow$
Single black pebble on v by assuming black pebbles on W $\Leftrightarrow$
$\mathbb{C}_t \cup \left\{\bigvee_{i=1}^{d} w_i \mid w \in W\right\} \vDash \bigvee_{i=1}^{d} v_i \Leftrightarrow$

For $d = 1$, we would like to get clause configuration–pebble correspondence:

$$\mathbb{C}_t = \begin{bmatrix} \overline{u}_1 \vee \overline{w}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee r_1 \\ v_1 \end{bmatrix}$$



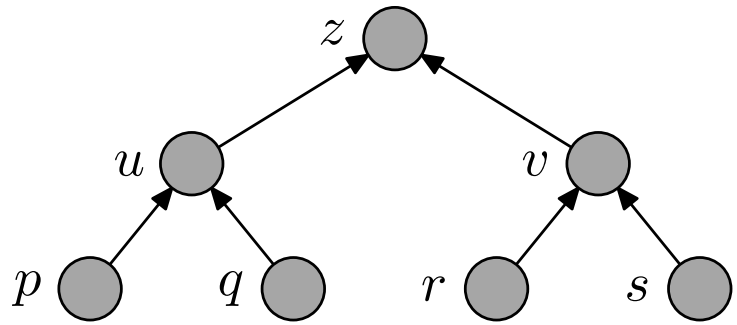
Induced Pebbles and Clause Configuration Size

- Formalizing this yields interpretation of clause configuration $\mathbb{C}_t$ derived from Peb_G^d in terms of pebbles on G
- Hope that resolution proof $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ will correspond to black-white pebbling $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ of G under this interpretation
- But to get lower bound on space from this we need to show that

$$\mathbb{C}_t \text{ induces many pebbles}$$
$$\Downarrow$$
$$\mathbb{C}_t \text{ contains many clauses}$$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

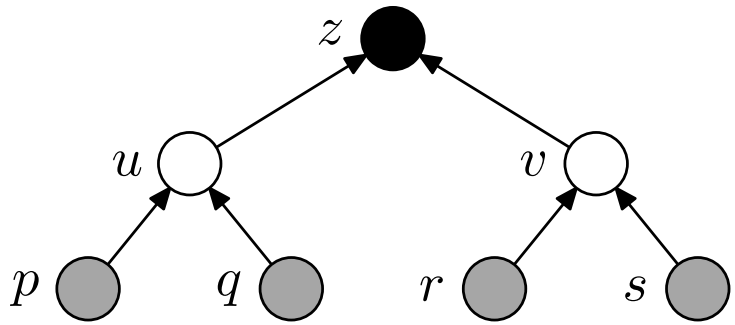


$\left[\quad\right]$

Empty start configuration

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

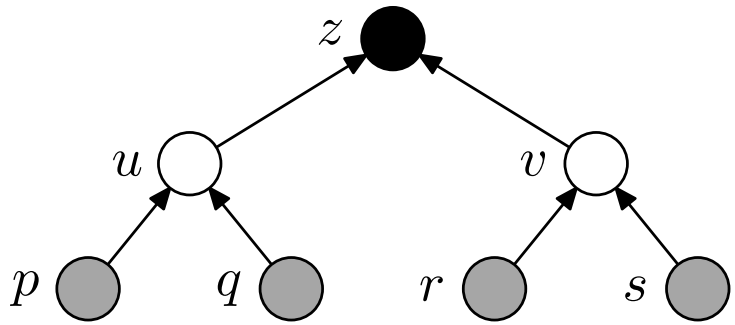


$$\left[\begin{array}{c} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \\ \\ \end{array}\right]$$

Download axiom 7: $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

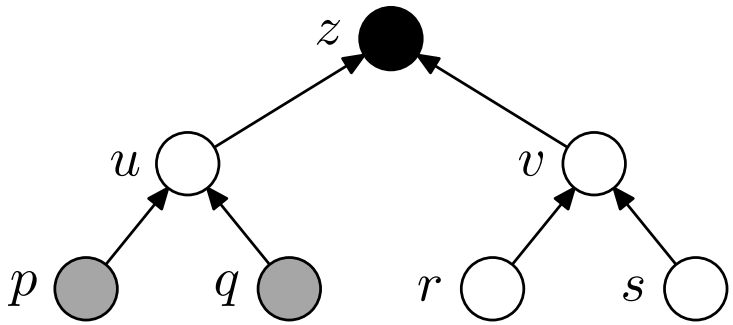


$$\left[\quad \overline{u}_1 \vee \overline{v}_1 \vee z_1 \quad \right]$$

Download axiom 7: $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

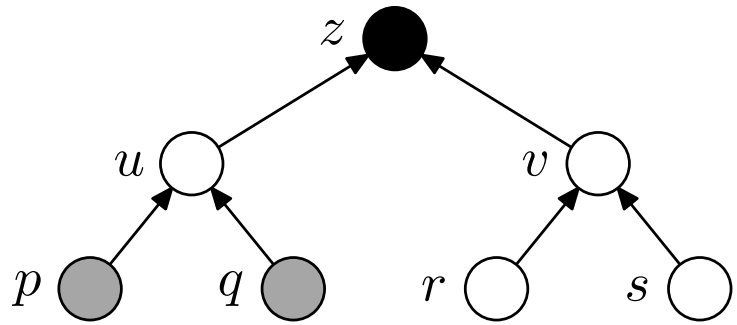


$$\begin{bmatrix} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \end{bmatrix}$$

Download axiom 6: $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

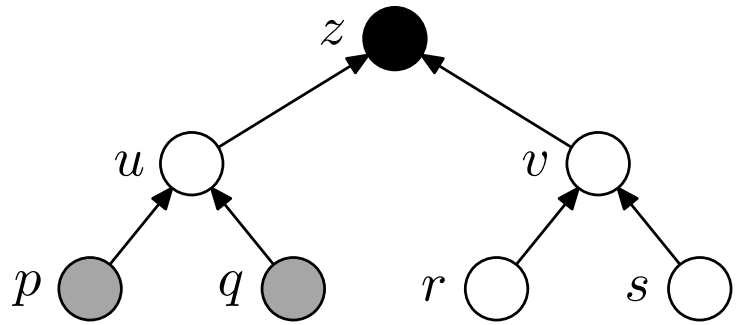


$$\begin{bmatrix} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \end{bmatrix}$$

Download axiom 6: $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

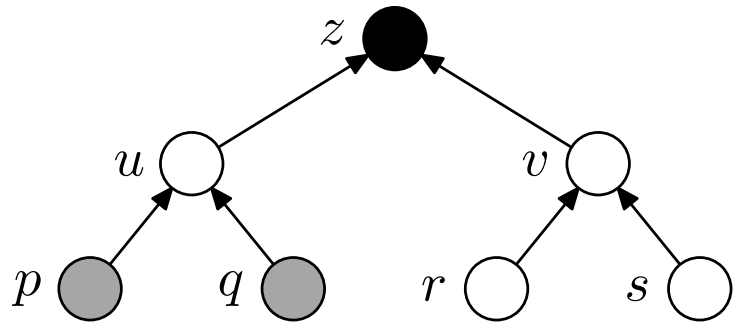


$$\left[\begin{array}{l} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \end{array}\right]$$

Infer $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$ from $\overline{r}_1 \vee \overline{s}_1 \vee v_1$ and $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

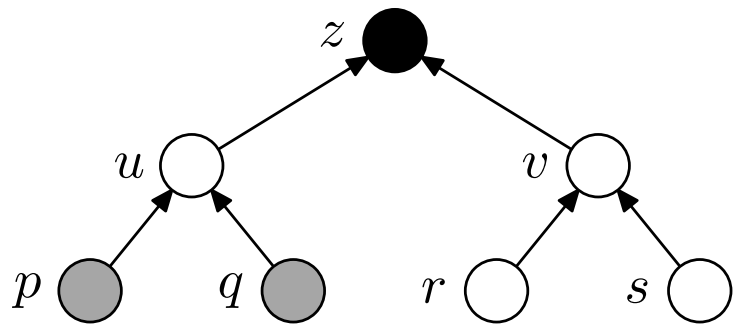


$$\begin{bmatrix} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$ from $\overline{r}_1 \vee \overline{s}_1 \vee v_1$ and $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

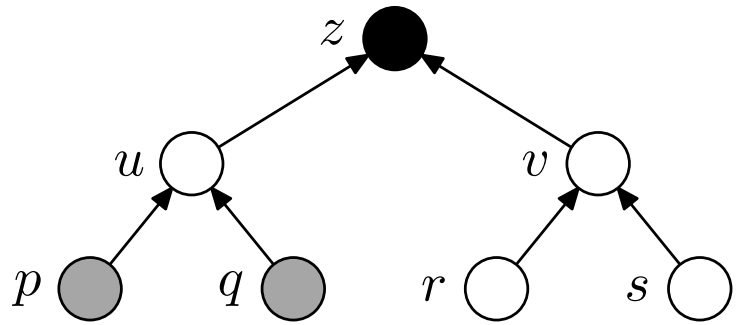


$$\begin{bmatrix} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$ from $\overline{r}_1 \vee \overline{s}_1 \vee v_1$ and $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

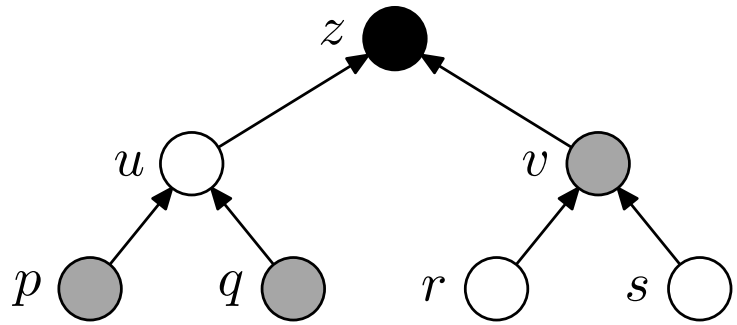


$$\begin{bmatrix} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

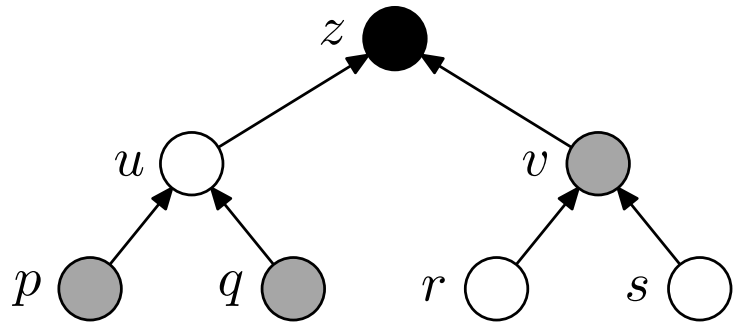


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

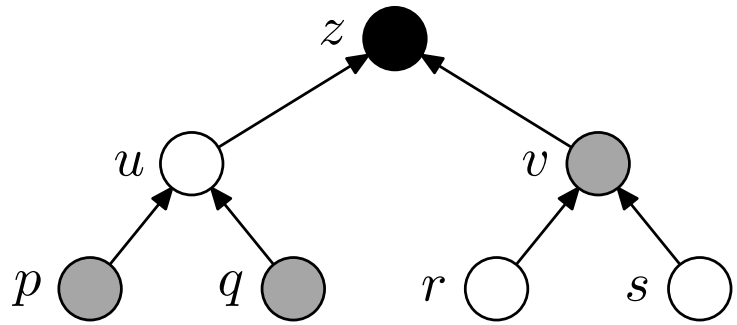


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \end{array}\right]$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

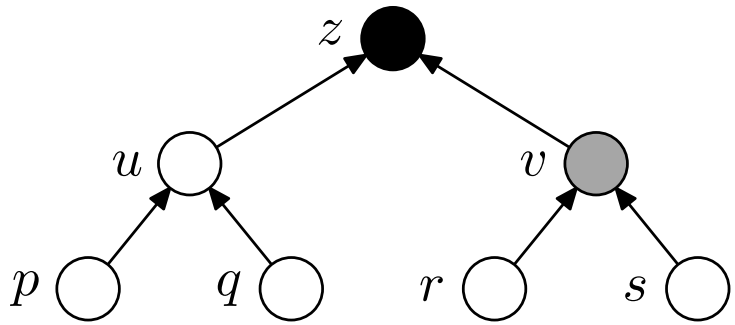


$$\left[\begin{array}{c} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \\ \end{array}\right]$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

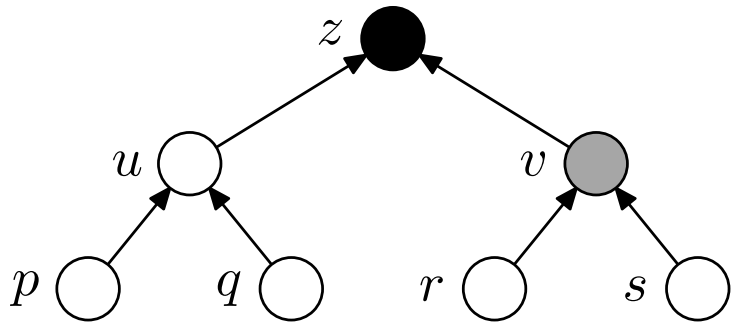


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \end{bmatrix}$$

Download axiom 5: $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

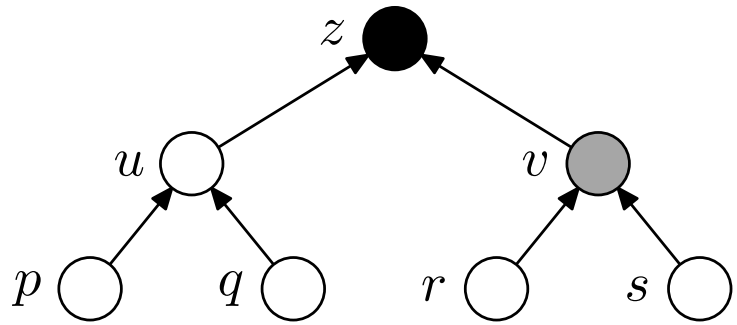


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \end{bmatrix}$$

Download axiom 5: $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

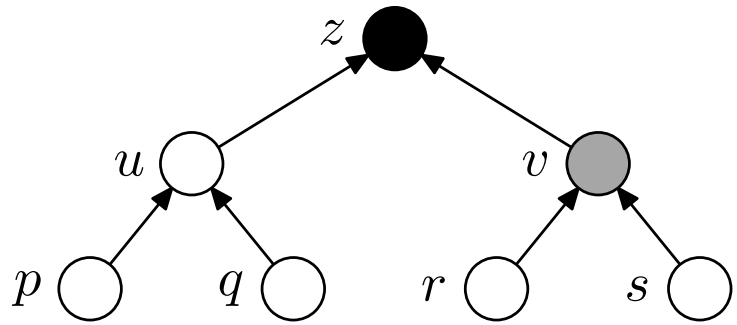


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \end{array}\right]$$

Infer $\overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$ from $\overline{p}_1 \vee \overline{q}_1 \vee u_1$ and $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

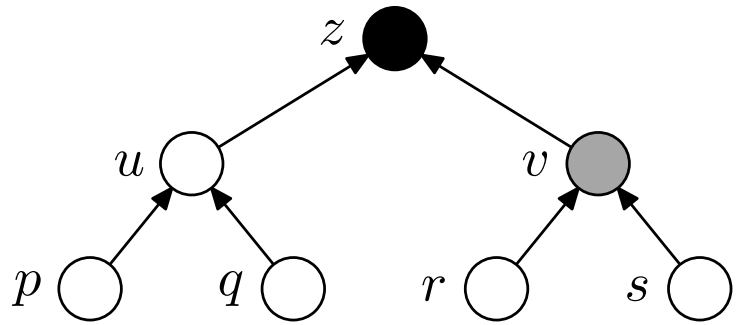


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$ from $\overline{p}_1 \vee \overline{q}_1 \vee u_1$ and $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

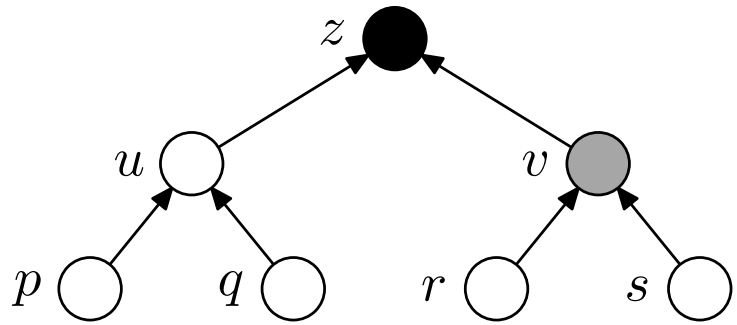


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$ from $\overline{p}_1 \vee \overline{q}_1 \vee u_1$ and $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

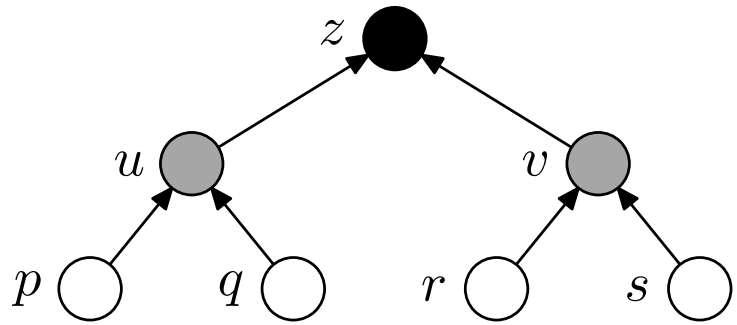


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

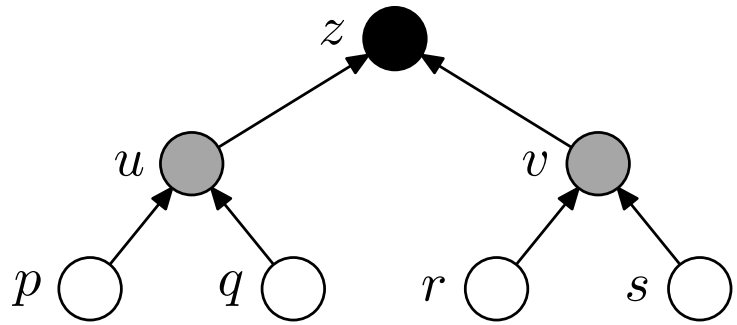


$$\begin{bmatrix} \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

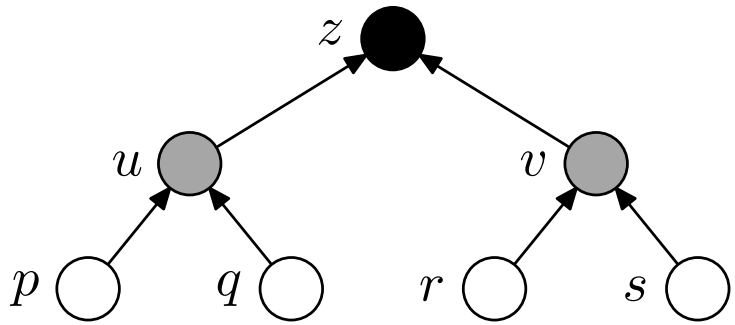


$$\left[\begin{array}{l} \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{array}\right]$$

Erase clause $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

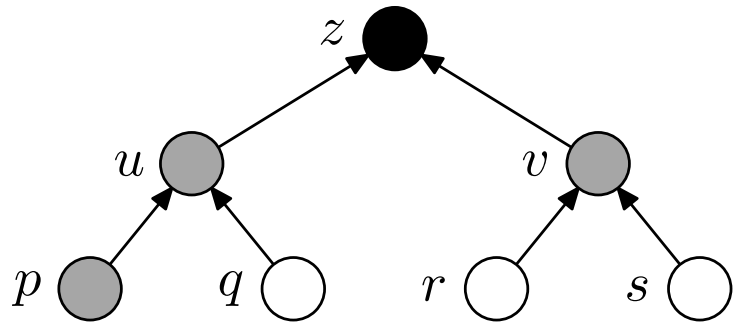


$$\left[\begin{array}{l} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \\ \end{array}\right]$$

Erase clause $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

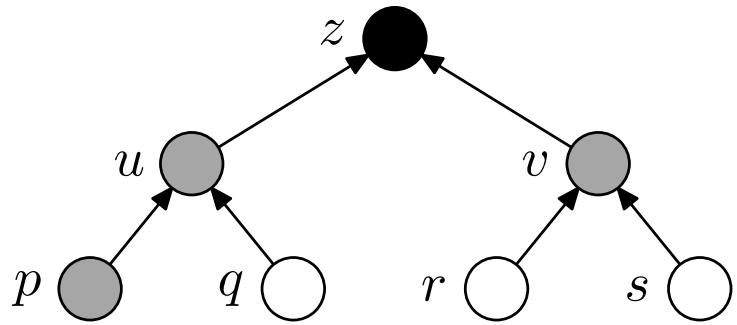


$$\begin{bmatrix} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ p_1 \end{bmatrix}$$

Download axiom 1: p_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

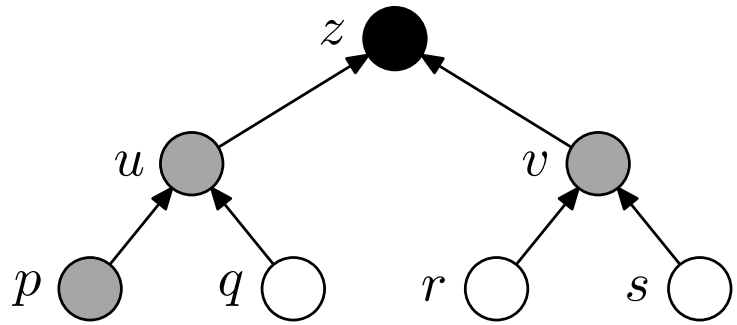


$$\begin{bmatrix} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ p_1 \\ \end{bmatrix}$$

Download axiom 1: p_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

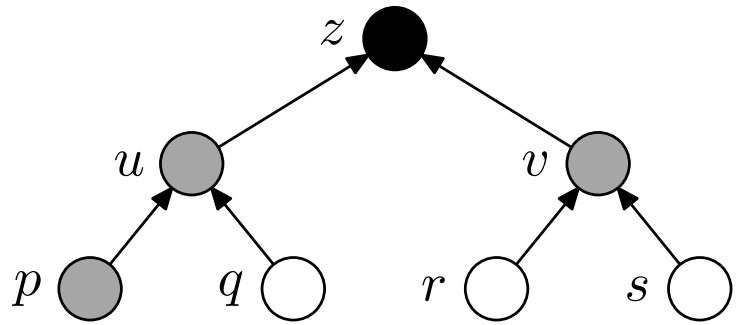


$$\begin{bmatrix} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ p_1 \end{bmatrix}$$

Infer $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$ from p_1 and $\overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

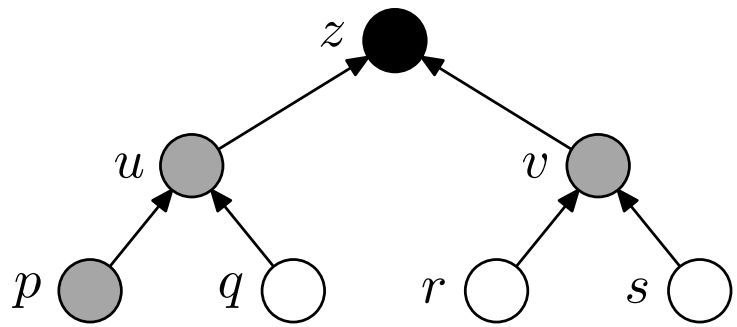


$$\begin{bmatrix} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ p_1 \\ \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$ from p_1 and $\overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

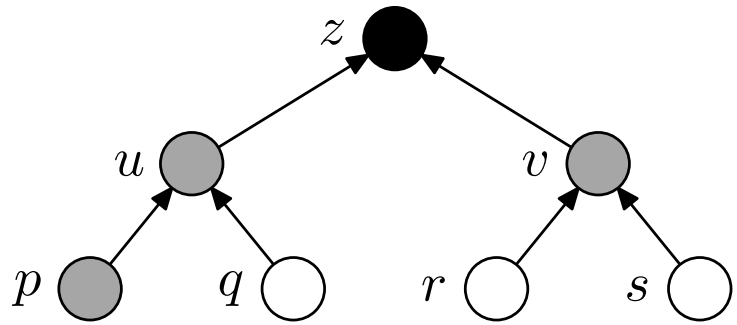


$$\begin{bmatrix} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ p_1 \\ \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$ from p_1 and $\overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

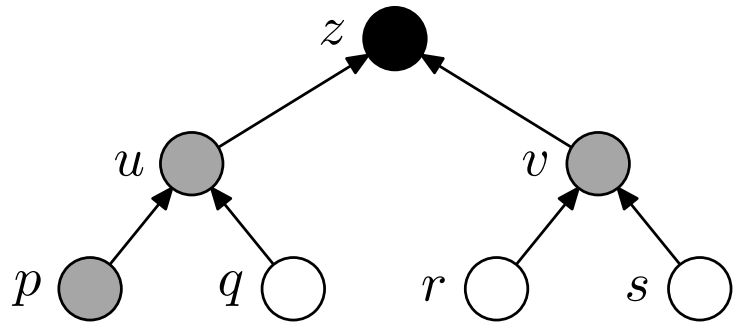


$$\begin{bmatrix} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ p_1 \\ \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause
$\overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target



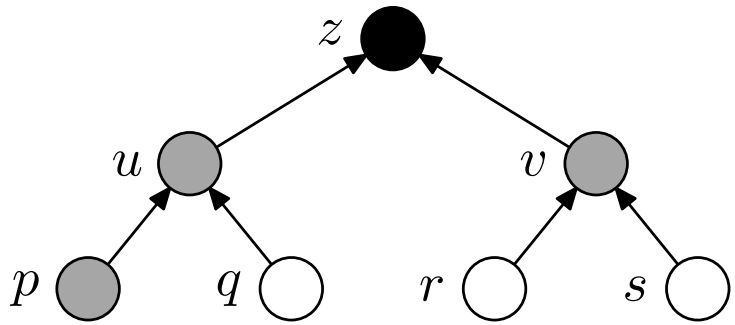
$$\left[\begin{array}{l} p_1 \\ \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \end{array}\right]$$

Erase clause

$\overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

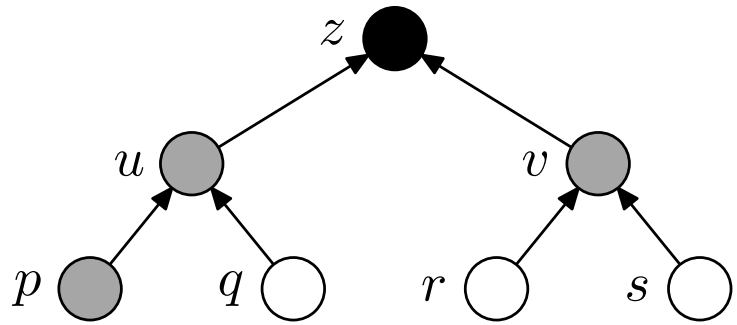


$$\left[\begin{array}{l} p_1 \\ \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{array}\right]$$

Erase clause p_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

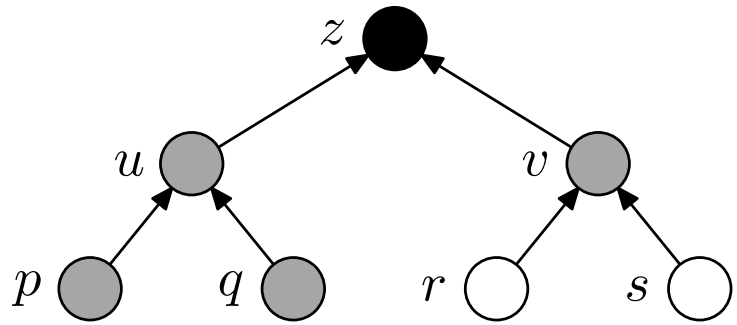


$$\left[\begin{array}{l} \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \\ \end{array}\right]$$

Erase clause p_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

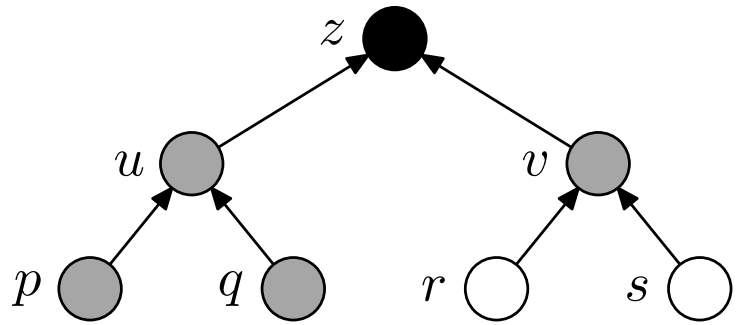


$$\begin{bmatrix} \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ q_1 \\ \\ \end{bmatrix}$$

Download axiom 2: q_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

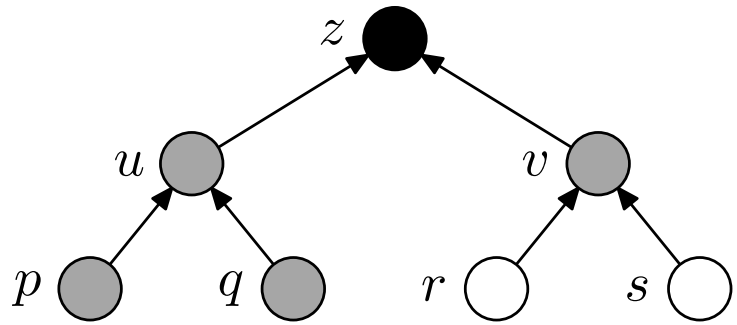


$$\begin{bmatrix} \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ q_1 \\ \\ \end{bmatrix}$$

Download axiom 2: q_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

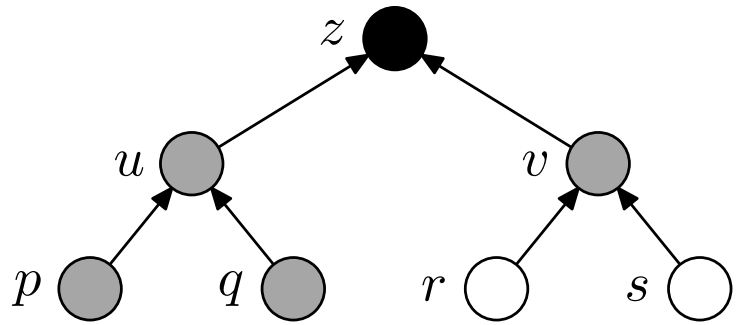


$$\left[\begin{array}{l} \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ q_1 \\ \\ \end{array}\right]$$

Infer $\overline{r}_1 \vee \overline{s}_1 \vee z_1$ from
q_1 and $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

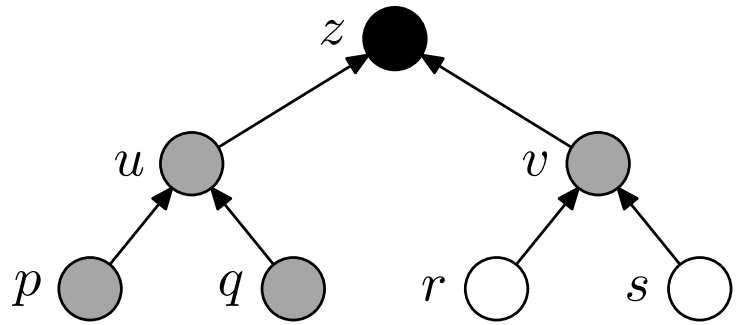


$$\begin{bmatrix} \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ q_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{r}_1 \vee \overline{s}_1 \vee z_1$ from
q_1 and $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

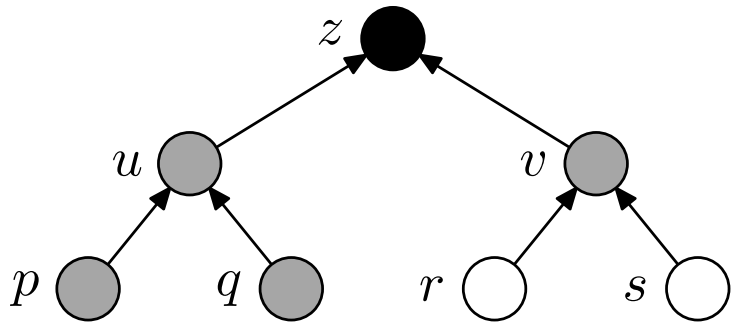


$$\begin{bmatrix} \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ q_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{r}_1 \vee \overline{s}_1 \vee z_1$ from
q_1 and $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

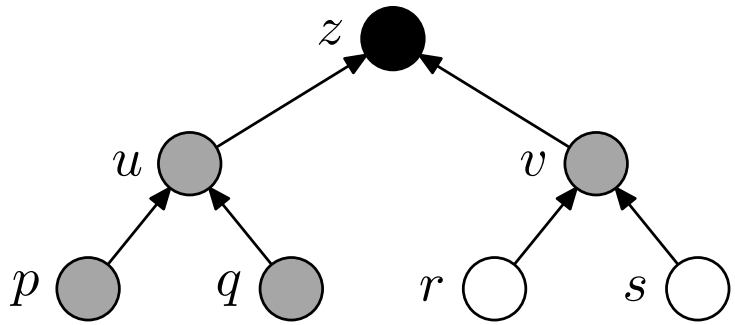


$$\begin{bmatrix} \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ q_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

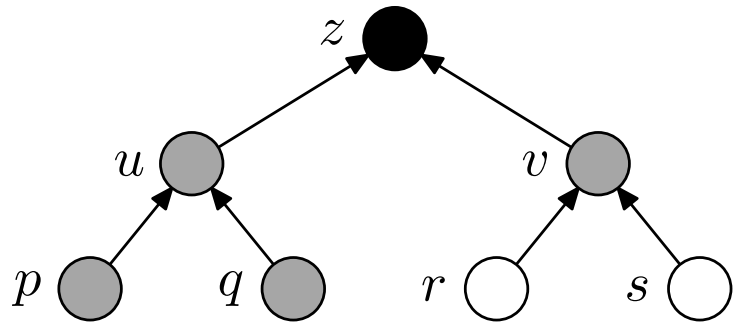


$$\begin{bmatrix} q_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

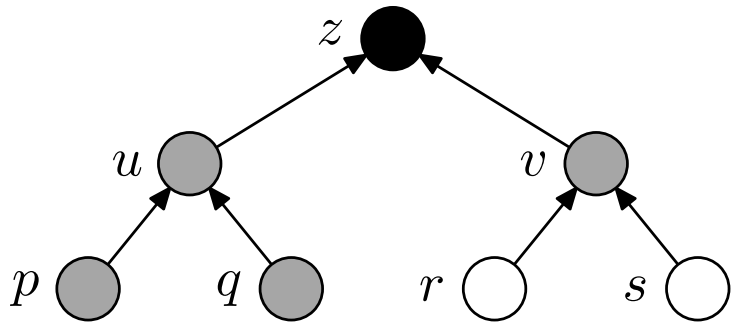


$$\left[\begin{array}{l} q_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{array}\right]$$

Erase clause q_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

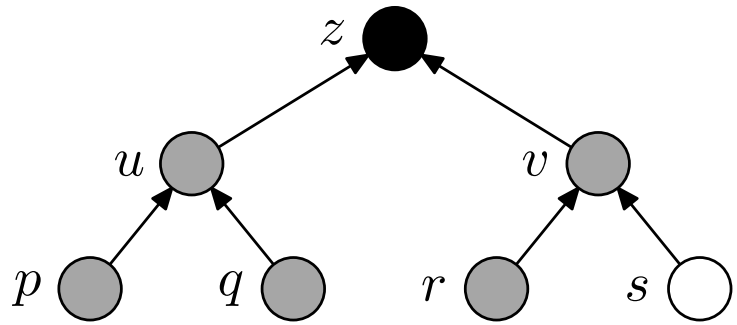


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \\ \\ \end{array}\right]$$

Erase clause q_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

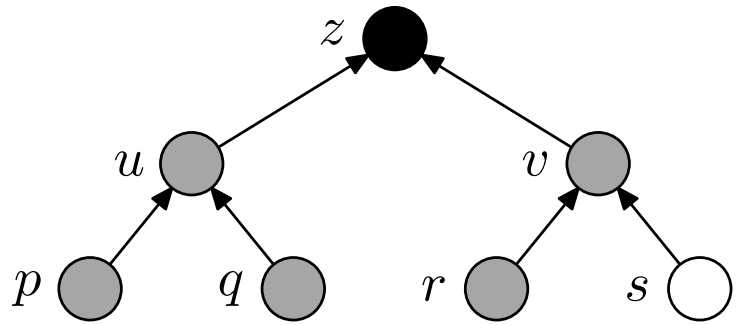


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \end{bmatrix}$$

Download axiom 3: r_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

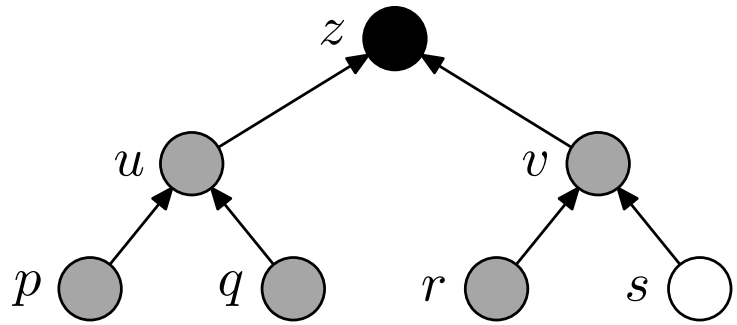


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \\ \end{bmatrix}$$

Download axiom 3: r_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

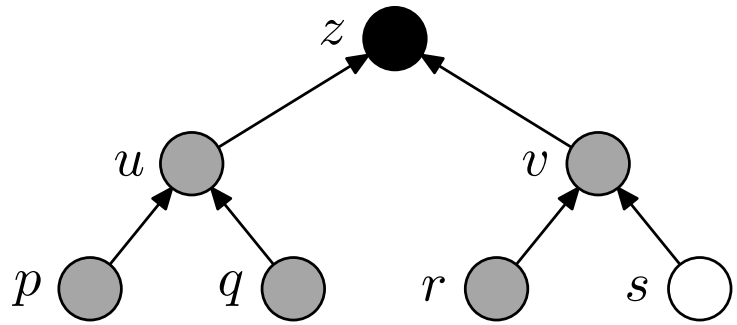


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \end{array}\right]$$

Infer $\overline{s}_1 \vee z_1$ from
r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

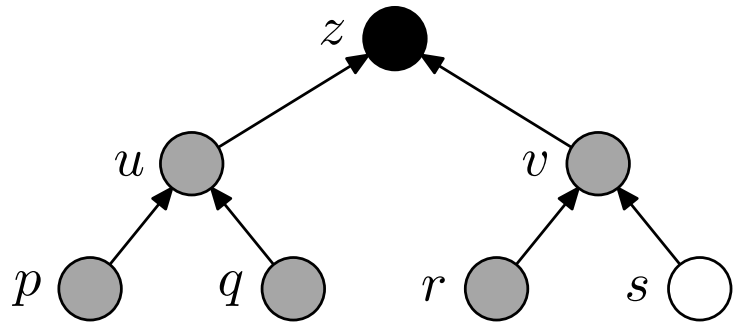


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{s}_1 \vee z_1$ from
r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

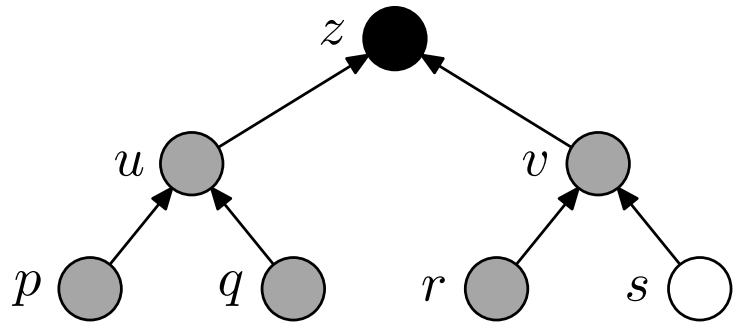


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{s}_1 \vee z_1$ from
r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

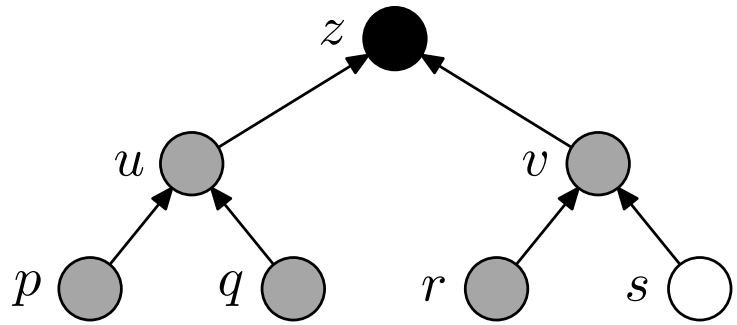


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \overline{s}_1 \vee z_1 \end{array}\right]$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

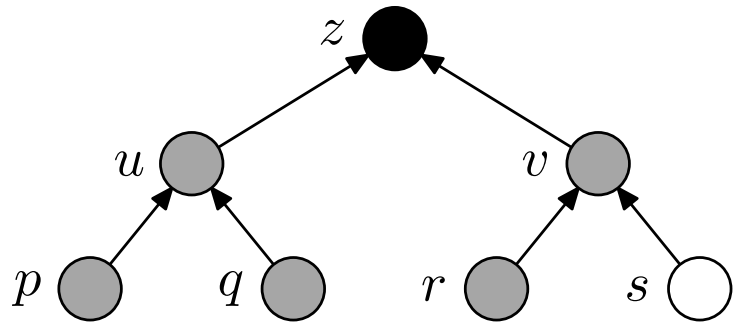


$$\begin{bmatrix} r_1 \\ \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

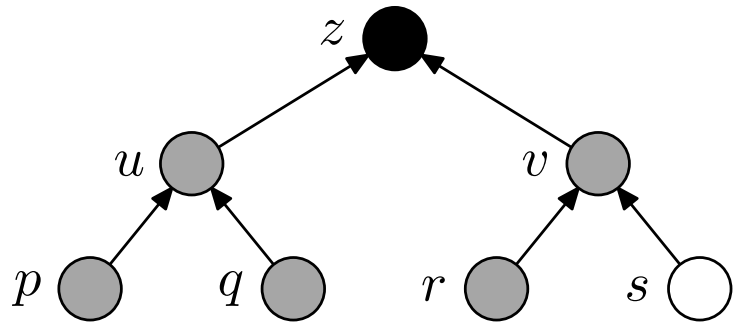


$$\begin{bmatrix} r_1 \\ \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause r_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

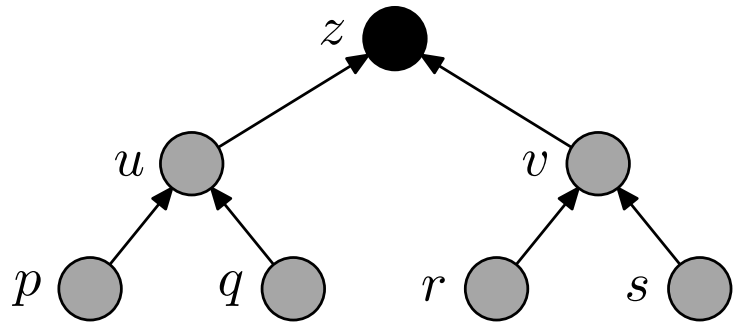


$$\left[\begin{array}{l} \overline{s}_1 \vee z_1 \\ \\ \end{array}\right]$$

Erase clause r_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

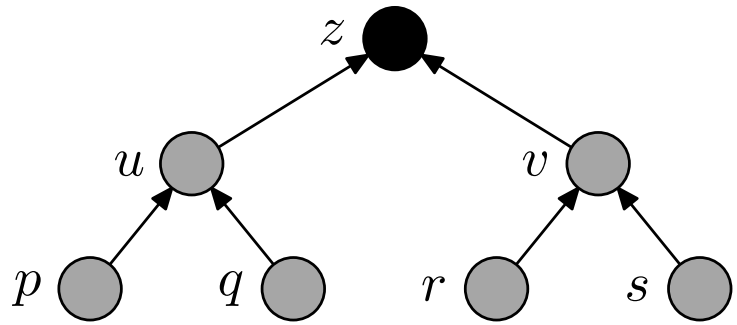


$$\begin{bmatrix} \overline{s}_1 \vee z_1 \\ s_1 \\ \\ \end{bmatrix}$$

Download axiom 4: s_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

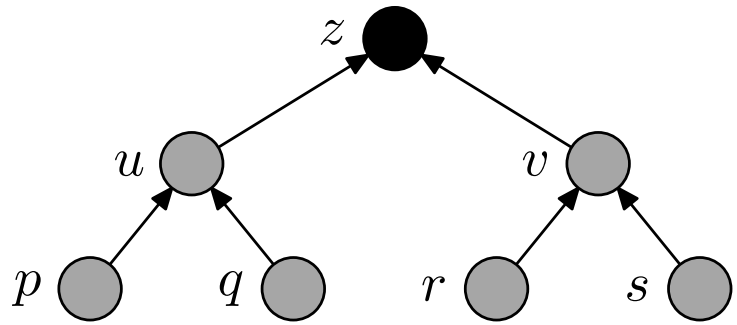


$$\begin{bmatrix} \overline{s}_1 \vee z_1 \\ s_1 \\ \end{bmatrix}$$

Download axiom 4: s_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

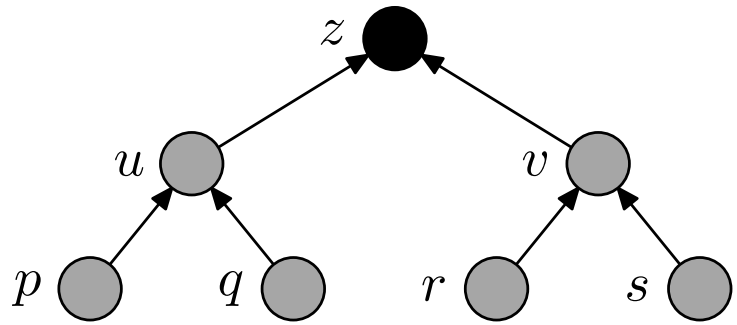


$$\begin{bmatrix} \overline{s}_1 \vee z_1 \\ s_1 \end{bmatrix}$$

Infer z_1 from
s_1 and $\overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

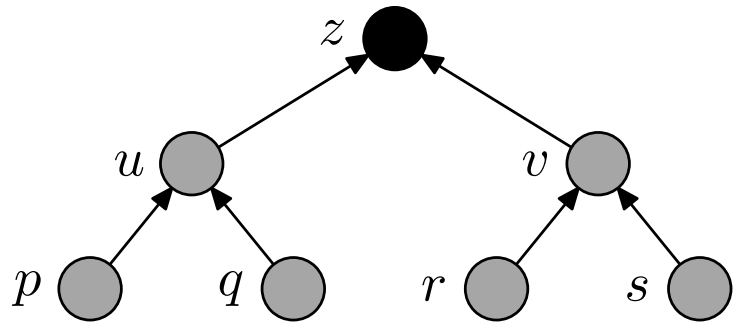


$$\begin{bmatrix} \overline{s}_1 \vee z_1 \\ s_1 \\ z_1 \end{bmatrix}$$

Infer z_1 from
s_1 and $\overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

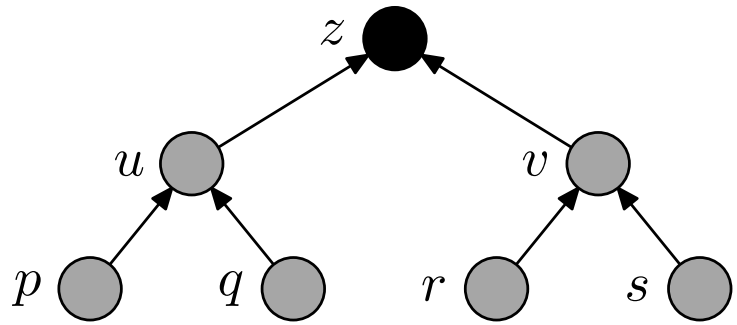


$$\begin{bmatrix} \overline{s}_1 \vee z_1 \\ s_1 \\ z_1 \end{bmatrix}$$

Infer z_1 from
s_1 and $\overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

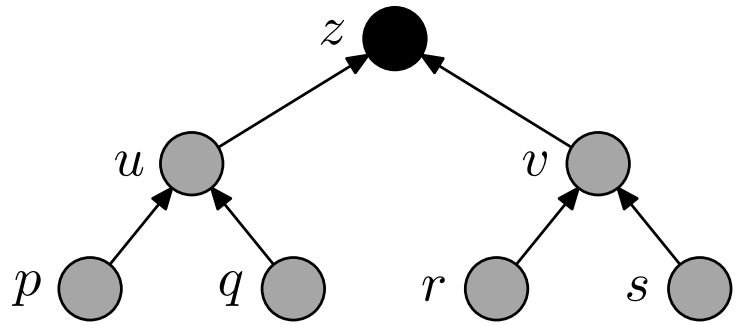


$$\begin{bmatrix} \overline{s}_1 \vee z_1 \\ s_1 \\ z_1 \end{bmatrix}$$

Erase clause $\overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

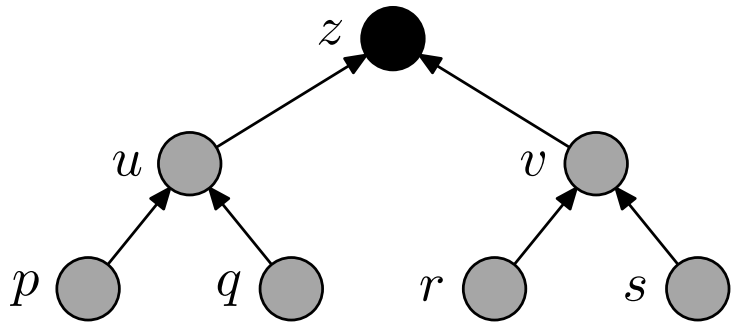


$$\begin{bmatrix} s_1 \\ z_1 \\ \end{bmatrix}$$

Erase clause $\overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

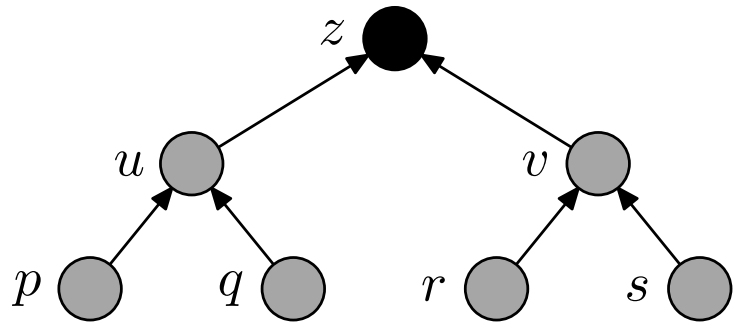


$$\begin{bmatrix} s_1 \\ z_1 \\ \\ \end{bmatrix}$$

Erase clause s_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

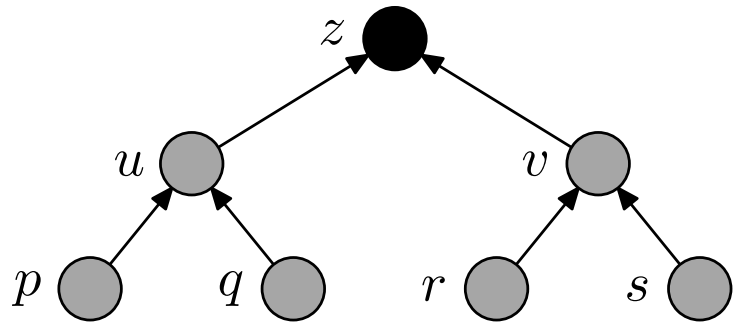


$$\begin{bmatrix} z_1 & \\ & \\ & \end{bmatrix}$$

Erase clause s_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

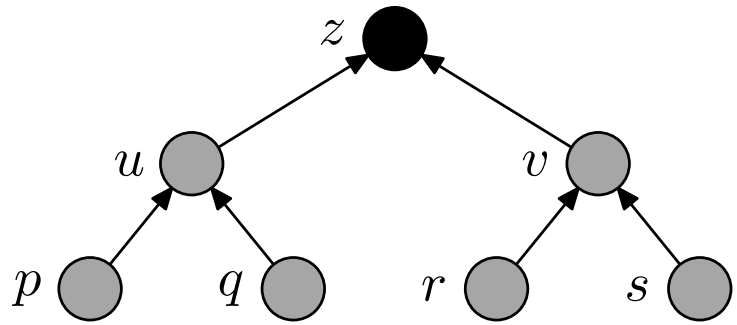


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \end{bmatrix}$$

Download axiom 8: $\overline{z}_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

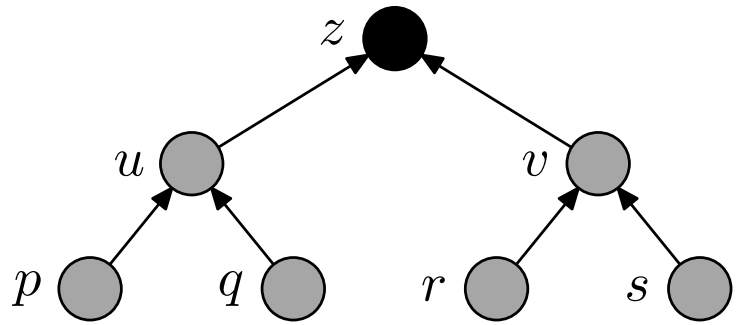


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \end{bmatrix}$$

Download axiom 8: $\overline{z}_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

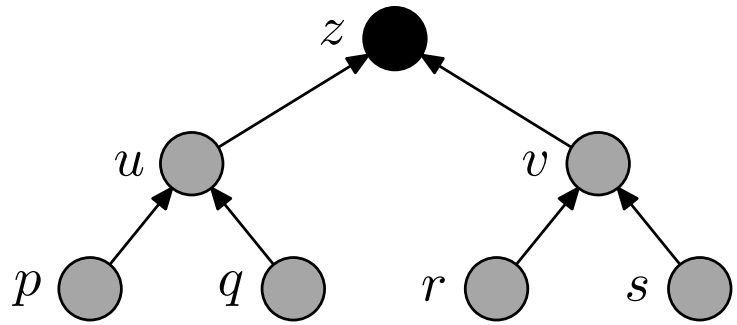


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \end{bmatrix}$$

Infer 0 from $\overline{z}_1$ and z_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

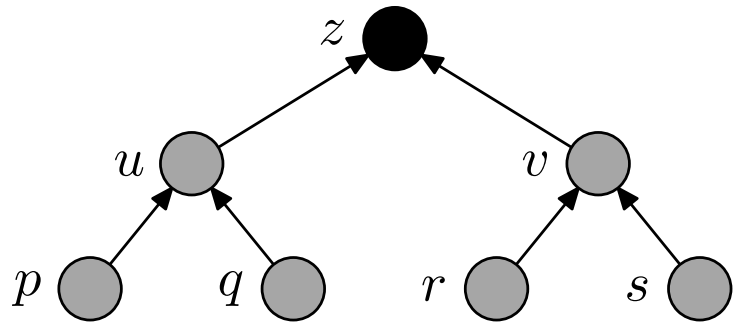


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ 0 \end{bmatrix}$$

Infer 0 from $\overline{z}_1$ and z_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target



$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ 0 \end{bmatrix}$$

Infer 0 from $\overline{z}_1$ and z_1

But Many Pebbles $\Rightarrow$ Many Clause for $d > 1$

This "top-down" proof in space 3 generalizes to any DAG G

- In terms of our induced pebble configurations:
  white pebbles are free for $d = 1$!
- In a sense, this is exactly why $Sp\left(Peb_G^1 \vdash 0\right) = \mathcal{O}(1)$
- But for $d > 1$ variables per vertex we can prove that
  # clauses $\geq$ # induced pebbles

Resolution Derivations Induce Modified Pebblings

Our interpretation of $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ as pebble configurations $\{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ yields a pebble game *of sorts*

- Erasures can lead to large blocks of pebbles suddenly disappearing—need to keep track of *exactly* which white pebbles have been used to get a black pebble on a vertex
- White pebbles can always be removed, but if so any black pebble dependent on these white pebbles must be removed as well
- "Backward" pebbling moves possible—white pebbles may slide upwards and black pebbles slide downwards (reversal moves)

Resolution Derivations Induce Modified Pebblings

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Resolution Derivations Induce Modified Pebblings

Our interpretation of $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ as pebble configurations $\{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ yields a pebble game *of sorts*

- Erasures can lead to large blocks of pebbles suddenly disappearing—need to keep track of *exactly* which white pebbles have been used to get a black pebble on a vertex
- White pebbles can always be removed, but if so any black pebble dependent on these white pebbles must be removed as well
- "Backward" pebbling moves possible—white pebbles may slide upwards and black pebbles slide downwards (reversal moves)

Pebbling Price in Labelled Pebble Game

Reversal moves might seem harmless

Move pebbles "in wrong direction" $\Rightarrow$
should be possible to eliminate without affecting pebbling price

- This intuition is wrong—reversal rule is fatal
- Destroys pebbling price for general graphs
  (for instance for pyramids)
- But for binary trees T_h we still have a pebbling price $\Omega(h)$

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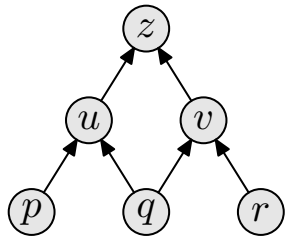

Main Theorem

Theorem

The space of refuting the pebbling contradiction of degree $d \geq 2$ over the complete binary tree of height h in resolution is $Sp(Peb_{T_h}^d \vdash 0) = \Theta(h)$.

Proof sketch.

- Upper bound easy (use "black-pebbling" resolution proof)
- For lower bound, let $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ be refutation of $Peb_{T_h}^d$ in minimal space
- Then there is some $\mathbb{C}_t \in \pi$ that induces $\Omega(h)$ pebbles in T_h
- Thus $Sp(\pi) \geq |\mathbb{C}_t| \geq$ # pebbles induced by $\mathbb{C}_t = \Omega(h)$. □

A Separation of Space and Width in Resolution

Corollary

For all $k \geq 4$, there is a family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and refutation space $Sp(F_n \vdash 0) = \Theta(\log n)$.

Proof.

We know $W(Peb_G^d \vdash 0) = \mathcal{O}(d)$ for all G.

Fix $d \geq 2$, let $F_n = Peb_{T_h}^d$ for $h = \lfloor \log(n+1) \rfloor$ and use the Main Theorem. □

Conclusion

- First lower bound on space in resolution which is not the consequence of a lower bound on width but instead separates the two measures
- Answers an open question in several previous papers
- We believe that it should be possible to strengthen this result in (at least) two ways

Open Problems

1. Generalize to arbitrary DAGs

Conjecture 1

For G an arbitrary DAG and $d \geq 2$ it holds that $Sp\left(Peb_G^d \vdash 0\right) = \Omega\left(BW\text{-}Peb(G)\right)$.

Would yield almost optimal separation $\Omega(n/\log n)$ between space and width—best conceivable is $\Omega(n)$

2. Generalize to k-DNF resolution proof systems $\mathcal{R}es(k)$ and prove space hierarchy

Conjecture 2

$Sp_{\mathcal{R}es(k+1)}\left(Peb_{T_h}^{k+1} \vdash 0\right) = \mathcal{O}(1)$ but $Sp_{\mathcal{R}es(k)}\left(Peb_{T_h}^{k+1} \vdash 0\right) = \Omega(h)$.

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References

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Extended abstract to appear in STOC '06

Thank you for your attention!