Certified Symmetry and Dominance Breaking for Combinatorial Optimisation

Jakob Nordström

University of Copenhagen
and Lund University

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Joint work with Bart Bogaerts, Stephan Gocht, and Ciaran McCreesh

Combinatorial Solving and Optimisation

- Revolution last couple of decades in combinatorial solvers for
  - Boolean satisfiability (SAT) solving [BHvMW21]*
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]

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- Software testing doesn't suffice to resolve this problem
- Formal verification techniques cannot deal with level of complexity of modern solvers

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Certified Results with Proof Logging

Solution: Design certifying algorithms [ABM+11, MMNS11] that

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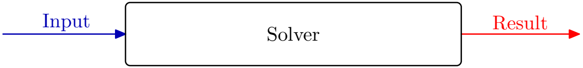

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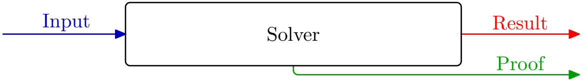

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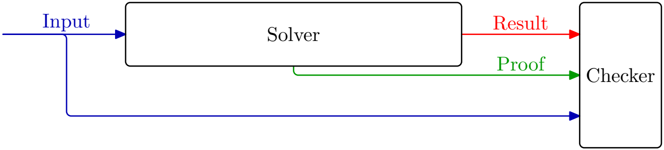

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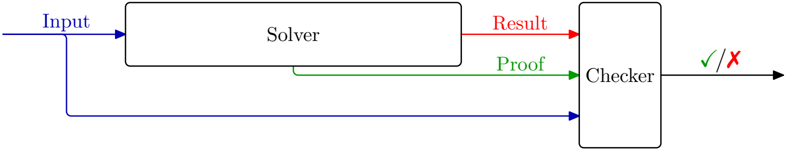

Workflow:

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3. Feed input + result + proof to proof checker
4. Verify that proof checker says result is correct

Yet Another SAT Success Story

Many proof logging formats for SAT solving using CNF clausal format:

- DRAT [HHW13a, HHW13b, WHH14]
- GRIT [CMS17]
- LRAT [CHH+17]
- ...

Well established — required in main track of SAT competitions
Crucial for unsatisfiable formulas

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But efficient proof logging has remained out of reach for stronger paradigms

And, in fact, even for some advanced SAT solving techniques:

- cardinality reasoning
- Gaussian elimination
- symmetry handling

Our Work: Efficient Proof Logging for Symmetry Breaking

Paper *Certified Symmetry and Dominance Breaking for Combinatorial Optimisation* at *AAAI '22* [BGMN22]:

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- Based on 0-1 integer linear constraints instead of clauses
- Uses cutting planes method [CCT87] with additional rules

Outline of Presentation

What I hope to cover in the rest of this presentation:

- Basics of proof logging with 0-1 linear constraints
- New rule for symmetry and dominance breaking
- Application to symmetry breaking for SAT solving (also other applications, but focus here on SAT)
- Some future research directions

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Caveat: Only exact problems in this talk but:

- This is already very challenging
- Ideas seem likely to generalize

0-1 Integer Linear (a.k.a. Pseudo-Boolean) Constraints

Pseudo-Boolean (PB) constraints are 0-1 integer linear constraints

$$C \doteq \sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Pseudo-Boolean formulas $F \doteq \bigwedge_{i=1}^{m} C_i$ are conjunctions of pseudo-Boolean constraints (a.k.a. 0-1 integer linear programs)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

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$$x \lor \overline{y} \lor z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

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3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i\ell_i \geq A \qquad \sum_i b_i\ell_i \geq B}{\sum_i(c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

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(See [BN21] for more details about cutting planes)

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- **Fact:** Fully sufficient for proof logging for so-called conflict-driven clause learning [BS97, MS99, MMZ$^+$01]
- Also need **extension** rule (analogue of RAT [JHB12] used in SAT proof logging) to deal with, e.g., preprocessing/presolving

Extension Rule: Redundance-Based Strengthening

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- Witness ω should be specified and implication efficiently verifiable by very simple checks (technical details omitted)

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Zoom tutorial on all of these developments Mon Nov 28 at 14:00 CET
Combinatorial Solving with Provably Correct Results
See `http://www.jakobnordstrom.se/miao-seminars`

The Challenge of Symmetries

(Syntactic) symmetry: substitution σ preserving F ($F{\restriction}_\sigma \doteq F$)

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- Can play crucial role in CP and MIP problems [AW13, GSVW14]

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Add constraints filtering out symmetric solutions [ASM06, DBBD16]

Symmetric learning in SAT
Allow to add all symmetric versions of learned constraint [DBB17]

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Optimisation Problems

Deal with symmetry breaking by switching focus to optimisation
(which the title of the talk kind of promised anyway)

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Pseudo-Boolean optimisation

Minimize $f = \sum_i w_i \ell_i$ (for $w_i \in \mathbb{N}^+$) subject to constraints in F

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Minimize $f = \sum_i w_i \ell_i$ (for $w_i \in \mathbb{N}^+$) subject to constraints in F

Proof of optimality:

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Spoiler alert:

For decision problem, nothing stops us from inventing objective function (like $\sum_{i=1}^{n} 2^{n-i} \cdot x_i$ minimized by lexicographic order)

Proof Logging for Optimisation Problems

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Rules must preserve (at least one) optimal solution

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Add constraint C to formula F if exists witness substitution ω such that

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- Applying ω should strictly decrease f
- If so, don't need to show that $C\restriction_\omega$ implied!

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8. Can't go on forever, so finally reach α' satisfying $F \wedge C$

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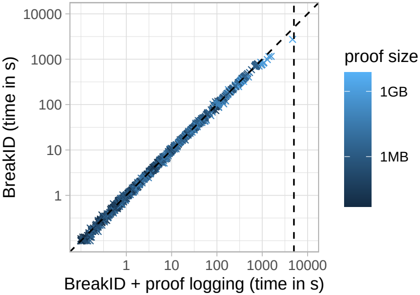
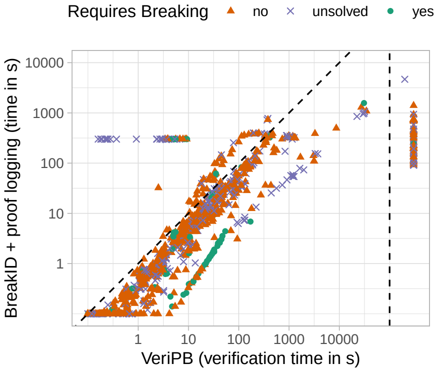
3. Derive CNF encoding of lex-leader constraints from PB constraint (in same spirit as PB-to-CNF translation in [GMNO22])

$$\begin{array}{ll} y_0 & \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j \\ \overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) & y_j \vee \overline{y}_{j-1} \vee \overline{x}_j \\ \overline{y}_j \vee y_{j-1} & y_j \vee \overline{y}_{j-1} \vee \sigma(x_j) \end{array}$$

Experimental Evaluation

- Evaluated on SAT competition benchmarks
- BREAKID [DBBD16, Bre] used to find and break symmetries





- Proof logging overhead negligible
- Verification at most 20 times slower than solving for 95% of instances

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
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And more. . .

- Lots of challenging problems and interesting ideas

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And more...

- Lots of challenging problems and interesting ideas
- We're hiring! Talk to me to join the proof logging revolution! ☺

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **This work:** Efficient proof logging for symmetry and dominance breaking using cutting planes proof system with extensions

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Thank you for your attention!

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