Time-Space Trade-offs in Proof Complexity (and SAT Solving?)

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Algorithms, Complexity and Machine Learning:
A Tribute to Kurt Mehlhorn
Chalmers University of Technology
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A Fundamental Theoretical Problem. . .

Problem

Given a propositional logic formula F, is it true no matter how we assign values to its variables?

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TAUTOLOGY: Fundamental problem in theoretical computer science ever since Stephen Cook's NP-completeness paper in 1971

(And significance realized much earlier — cf. Gödel's letter 1956)

These days recognized as one of the main challenges for all of mathematics as identified by the Clay Mathematics Institute

Widely believed intractable in worst case — deciding whether this is so is one of the famous million dollar Millennium Problems

. . . with Huge Practical Implications

- All known algorithms run in exponential time in worst case
- But enormous progress on applied computer programs last 10–20 years (with important contributions from Chalmers)
- These so-called SAT solvers routinely deployed to solve large-scale real-world problems with millions of variables
- Used in e.g. hardware verification, software testing, software package management, artificial intelligence, cryptography, bioinformatics, and more
- But we also know small formulas with only about a hundred variables that trip up even state-of-the-art SAT solvers

Theoretical Understanding of SAT Solver Performance?

- Best known algorithms based on simple DPLL method (Davis-Putnam-Logemann-Loveland) from early 1960s extended with conflict-driven clause learning (CDCL)
- Can we gain better theoretical understanding of potential and limitations of CDCL SAT solvers?
- Key concerns in SAT solving: time and memory management
- What are the connections between these resources? Are they correlated? Are there trade-offs?
- This talk: What can the field of proof complexity say about these questions?

Tautologies and CNF Formulas

Conjunctive normal form (CNF)

ANDs of ORs of variables or negated variables
(or conjunctions of disjunctive clauses over literals)

Example:

$$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$$

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Proving that a formula in propositional logic is **always** satisfied

$\Updownarrow$

Proving that a CNF formula is **never** satisfied

I.e., evaluates to false however you set the variables

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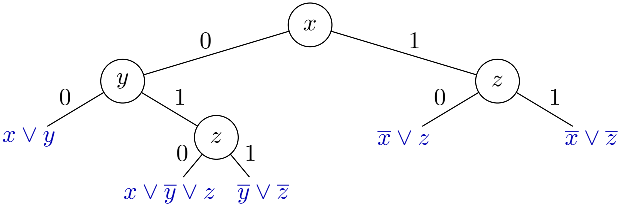
I.e., evaluates to false however you set the variables

(Sidenote: Can assume k-CNF — all clauses of constant size $\leq k$)

A Very Simplified Description of DPLL

Visualize execution of DPLL algorithm as search tree

- Branch on variable assignments in internal nodes
- Stop in leaves when falsfied clause found



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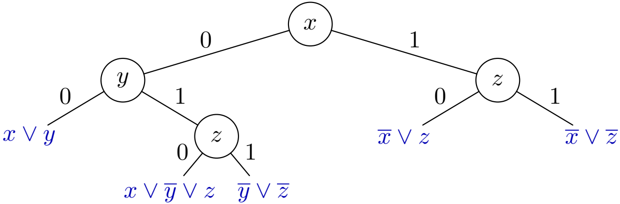

Many more ingredients in modern SAT solvers, for instance:

- Choice of branching variables crucial
- In leaf, compute & add reason for failure (clause learning)
- Restart every once in a while (but save computed info)

The Resolution Proof System

Resolution rule:

$$\frac{B \lor x \quad C \lor \overline{x}}{B \lor C}$$

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Observation

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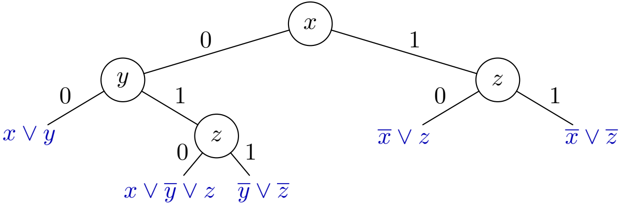
Prove F unsatisfiable by deriving the unsatisfiable empty clause $\bot$ from F by resolution

Proof of unsatisfiability = Refutation of formula
Will use terms "proof" and "refutation" as synonyms

DPLL and Resolution

A DPLL execution is essentially a resolution proof

Look at our example again:



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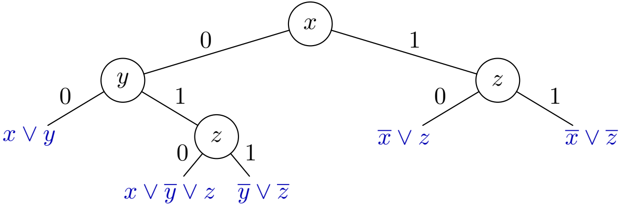

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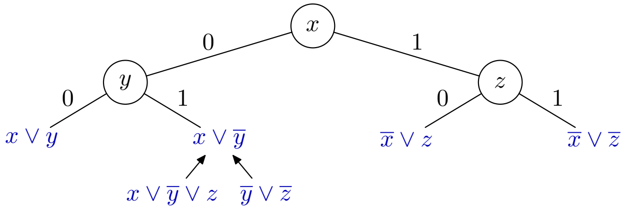

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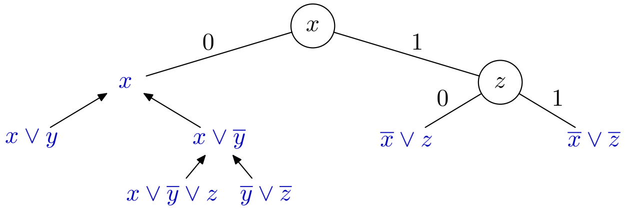

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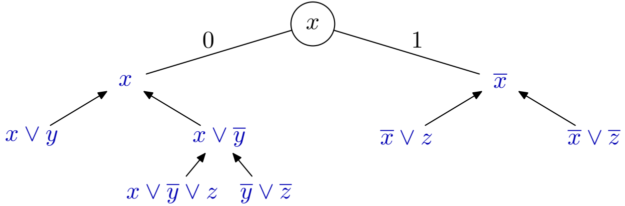

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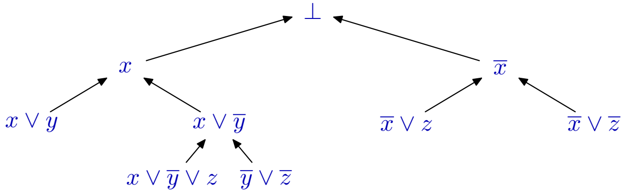

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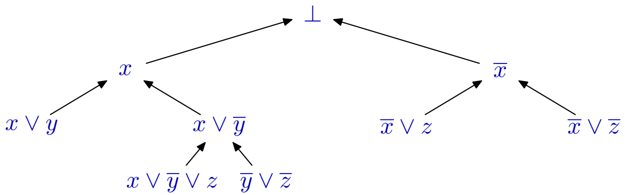

and apply resolution rule bottom-up

Holds also for clause learning — makes tree into a DAG

The Formal Model

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

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Can represent refutation as

- annotated list or
- DAG

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
4.	$\overline{y} \lor \overline{z}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2, 4)$
7.	x	Res$(1, 6)$
8.	$\overline{x}$	Res$(3, 5)$
9.	$\perp$	Res$(7, 8)$

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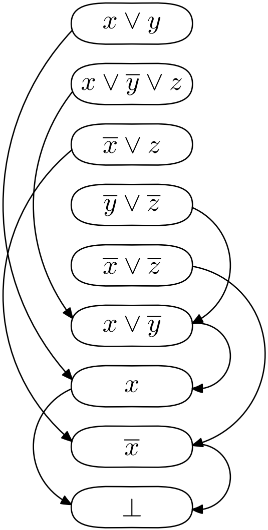

Resolution Size/Length

Let N = size of formula (total # literals)

Size/length = # clauses in refutation

Most fundamental measure in proof complexity

Lower bound on CDCL running time
(can extract resolution proof from execution trace)

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds in e.g. [Urq87, CS88, BW01]

Resolution Space

Space = max # clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — focus here on most common measure clause space

Space at step t: # clauses at steps $\leq t$ used at steps $\geq t$

1.	$x \vee y$	Axiom
2.	$x \vee \overline{y} \vee z$	Axiom
3.	$\overline{x} \vee z$	Axiom
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Example: Space at step 7 ...

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
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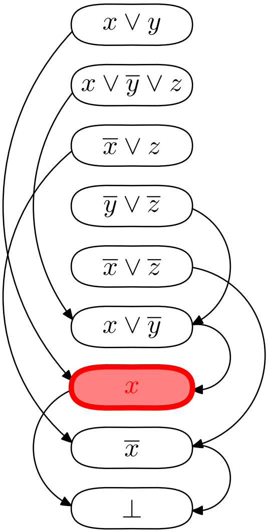

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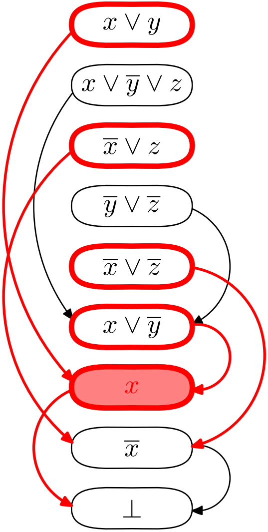
Space at step t: # clauses at steps $\leq t$ used at steps $\geq t$

Example: Space at step 7 is 5



Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ [ET01]

- Build search tree of depth $\leq N$
- Derive root clause of one subtree
- Keep in memory while doing other subtree; then resolve
- $\#$ clauses needed in memory scales like tree height

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Two comments/questions:

- Lower bounds hold even for "magic algorithms" making optimal choices — maybe much stronger in practice?
- Linear upper bounds hold for exponential-size proofs — what about space for reasonably-sized proofs?

Comparing Size and Space

Some “rescaling” needed to get meaningful comparisons of size and space

- Size exponential in formula size in worst case
- Space at most linear
- So natural to compare space to logarithm of size

Size-Space Correlations?

$\exists$ constant space refutation $\Rightarrow$ $\exists$ polynomial size refutation [AD03]

What about other direction — does small size imply small space?

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What about other direction — does small size imply small space?
No, false in strongest sense possible

Theorem ([BN08])

There are k-CNF formula families of size N

- *refutable in size $\mathcal{O}(N)$*
- *requiring space $\Omega(N/\log N)$*

Optimal separation — given proof size $\mathcal{O}(N)$, always possible to achieve proof space $\mathcal{O}(N/\log N)$

Size-Space Trade-offs

Can also show collection of size-space trade-off results

Formulas are simple and explicit

Theorem ((informal) [BN11])

There are k-CNF formulas for which

- *exist resolution refutations in small size*
- *exist resolution refutations in small space*
- *optimization of one measure* causes *dramatic blow-up for other measure*

So no meaningful simultaneous optimization possible for size and space in the worst case

How to Get a Handle on Time-Space Relations?

Questions about time-space trade-offs fundamental in theoretical computer science

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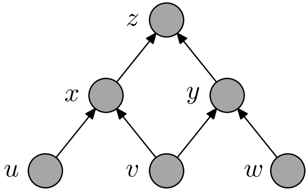
In particular, well-studied (and well-understood) for pebble games modelling calculations described by directed acyclic graphs ([CS76] and many others)

- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



# moves	0
Current # pebbles	0
Max # pebbles so far	0

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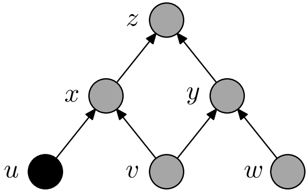

# moves	1
Current # pebbles	1
Max # pebbles so far	1

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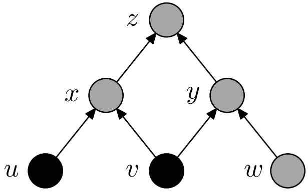

# moves	2
Current # pebbles	2
Max # pebbles so far	2

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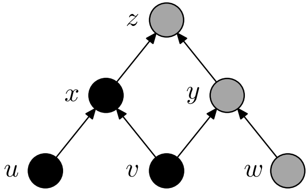

# moves	3
Current # pebbles	3
Max # pebbles so far	3

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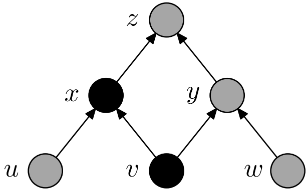

# moves	4
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex

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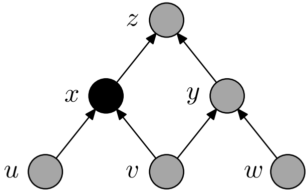

# moves	5
Current # pebbles	1
Max # pebbles so far	3

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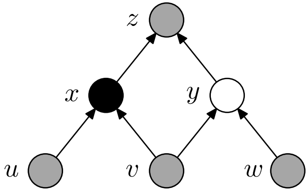

# moves	6
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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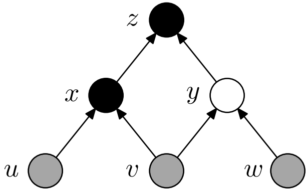
Goal: get single black pebble on sink vertex z of G



# moves	7
Current # pebbles	3
Max # pebbles so far	3

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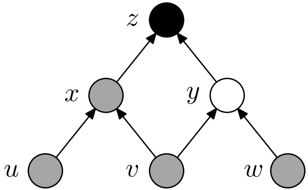

# moves	8
Current # pebbles	2
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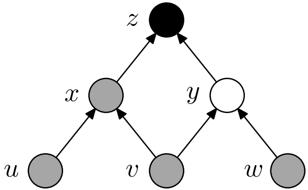

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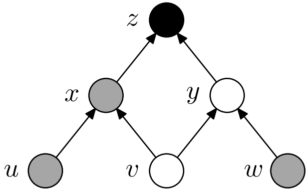

# moves	9
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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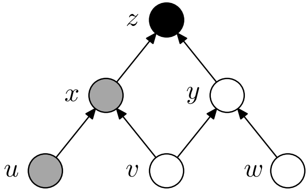

# moves	10
Current # pebbles	4
Max # pebbles so far	4

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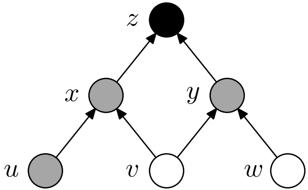

# moves	11
Current # pebbles	3
Max # pebbles so far	4

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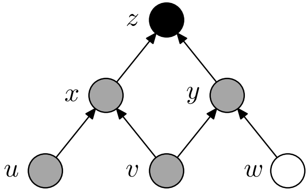

# moves	12
Current # pebbles	2
Max # pebbles so far	4

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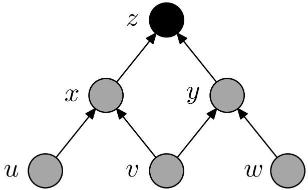

# moves	13
Current # pebbles	1
Max # pebbles so far	4

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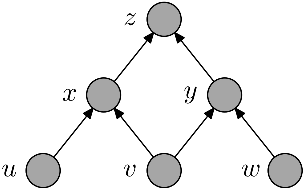
Pebbling Contradiction

CNF formula encoding pebble game on DAG G

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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Pebbling Contradiction

CNF formula encoding pebble game on DAG G

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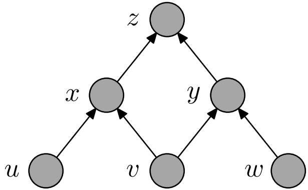

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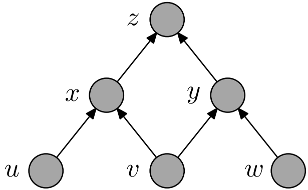

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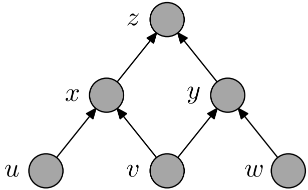

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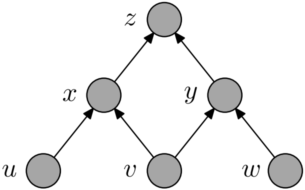

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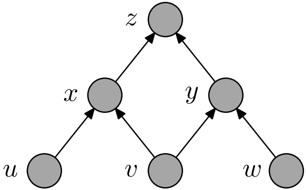

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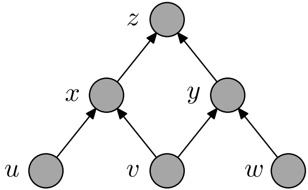

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Hope that pebbling properties of DAGs somehow carry over to resolution refutations of pebbling contradictions

A Problem and a Fix: Variable Substitution

Problem: Pebbling contradictions supereasy (solved by unit propagation) — no nontrivial lower bounds possible

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$$\overline{x} \lor z$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \lor (z_1 \oplus z_2)$$

$$\Downarrow$$

$$\begin{aligned} & (x_1 \lor \overline{x}_2 \lor z_1 \lor z_2) \\ \land\ & (x_1 \lor \overline{x}_2 \lor \overline{z}_1 \lor \overline{z}_2) \\ \land\ & (\overline{x}_1 \lor x_2 \lor z_1 \lor z_2) \\ \land\ & (\overline{x}_1 \lor x_2 \lor \overline{z}_1 \lor \overline{z}_2) \end{aligned}$$

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Now CNF formula inherits pebbling graph properties!

Trade-offs in the Superlinear Space Regime?

But...

- Pebbling contradictions always refutable in linear size and linear space simultaneously
- If exists small proof, always possible to find in linear space?
- Or are there formulas for which small proofs require superlinear space?

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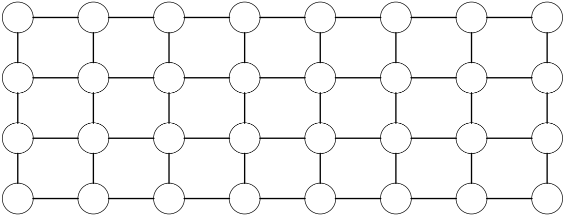
Theorem (informal [BBI12, BNT13])

For every $s \in \mathbb{N}^+$ there are k-CNF formulas for which

- *exist small proofs in size $N^{s+\mathcal{O}(1)}$ and space $N^{s+\mathcal{O}(1)}$*
- *exist space-efficient proofs in space $\mathcal{O}\left(s \log^2 N\right)$*
- *any proof in space $\mathcal{O}\left(N^{s/2}\right)$ requires superpolynomial size*

Tseitin Formulas over Long, Skinny Grids

- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$



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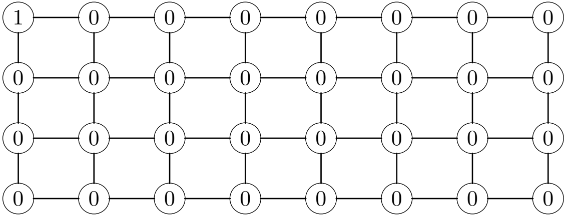

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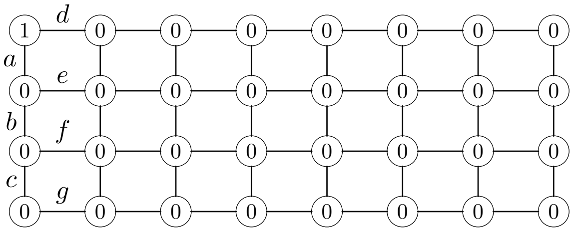

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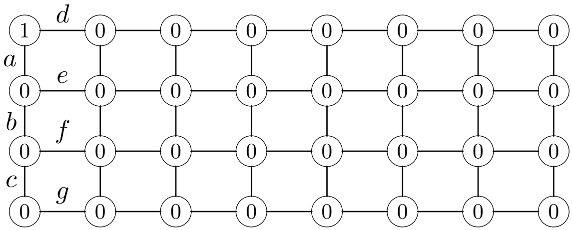

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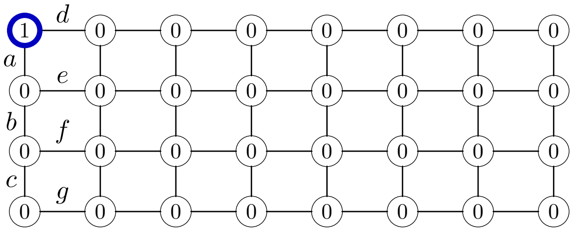
$(a \lor d)$
$\land\ (\overline{a} \lor \overline{d})$



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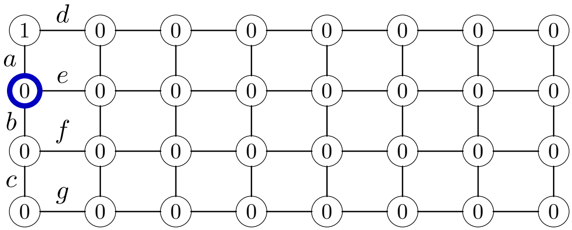

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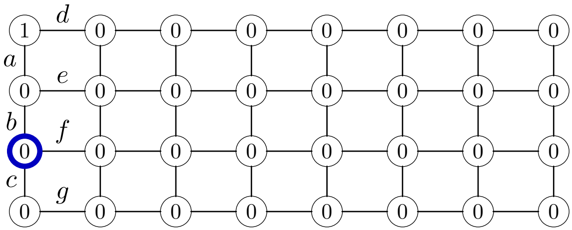

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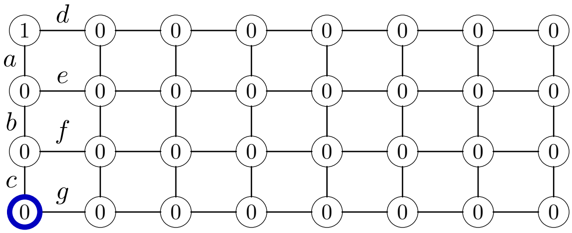

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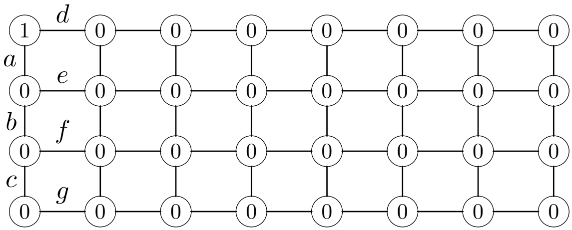
- Unsatisfiable — every edge counted twice, so total sum can't be odd



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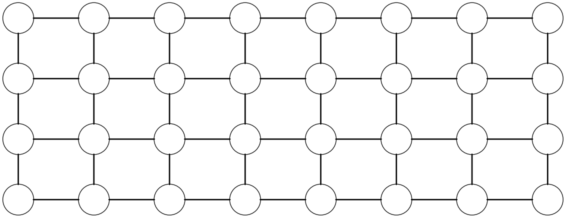
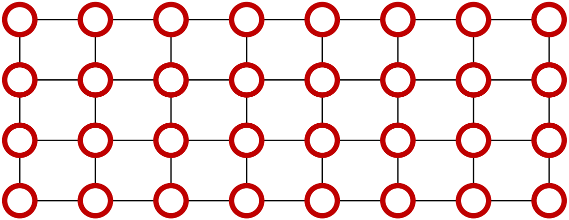
Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges



Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges
- Identify odd-charge component



Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges
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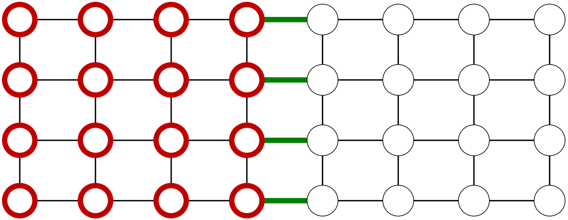
- Disconnect into two pieces by querying edges; then recurse



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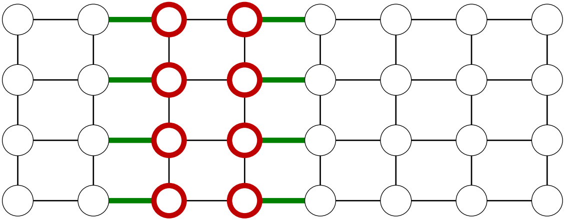

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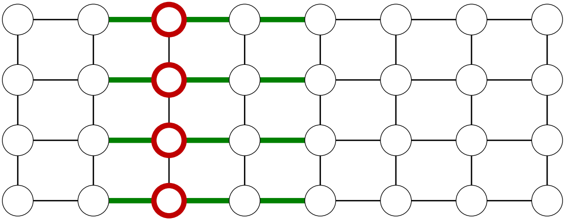

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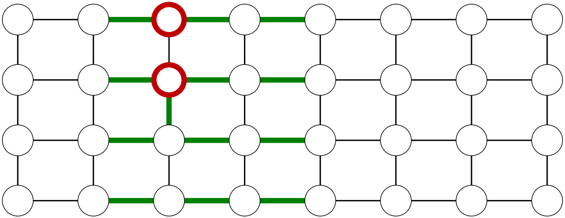

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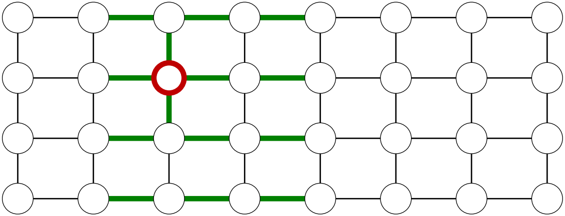

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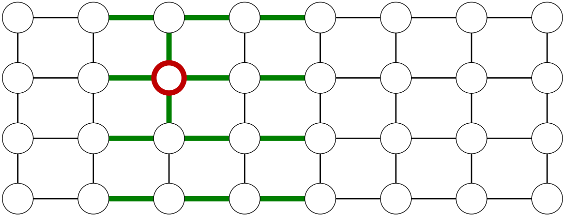
- Height of tree = proof space = $w \log m$
  (very space-efficient, but proof size exponential in space)



Small-Size "Dynamic programming" Proof

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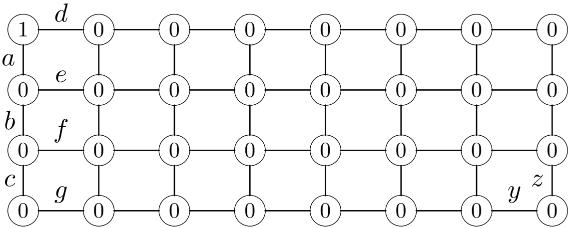

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- View constraints as linear equations $\bmod 2$
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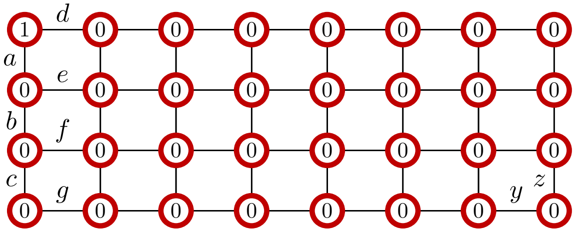

Small-Size "Dynamic programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
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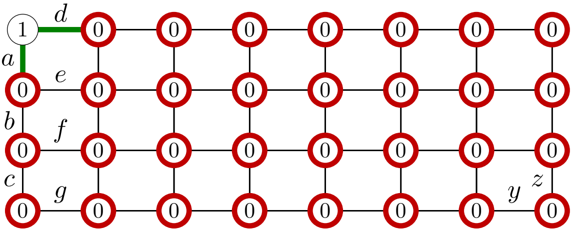
  But parity of $w + 1$ variables $\Rightarrow 2^w$ clauses

$a + d = 1$



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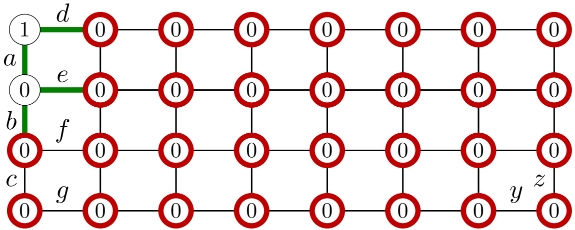

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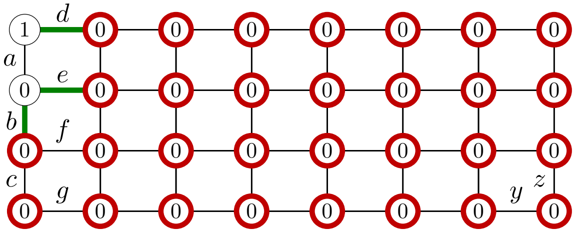

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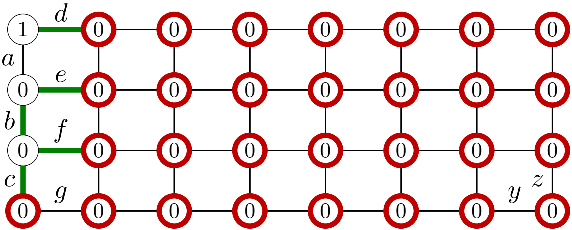

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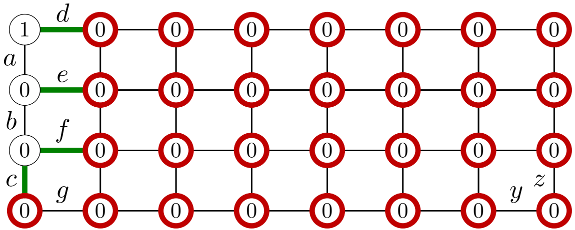

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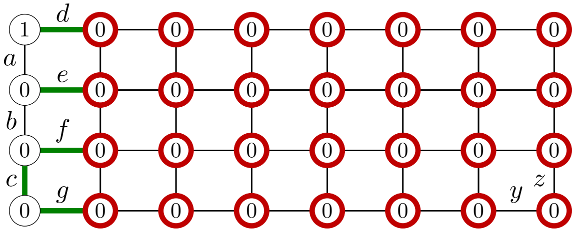

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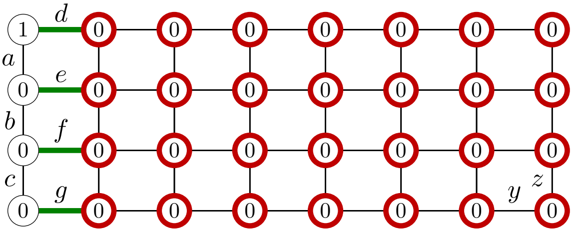

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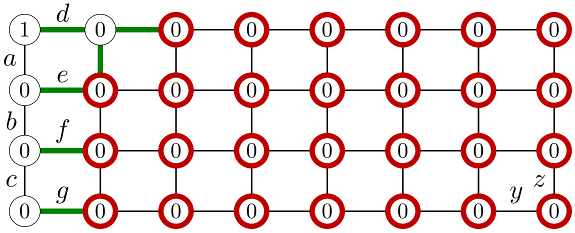

Small-Size "Dynamic programming" Proof

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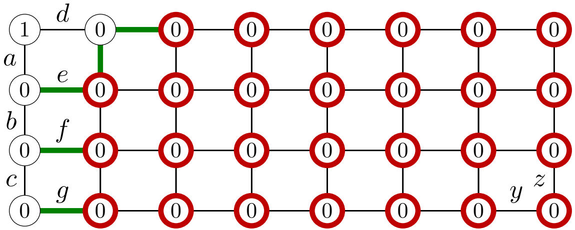

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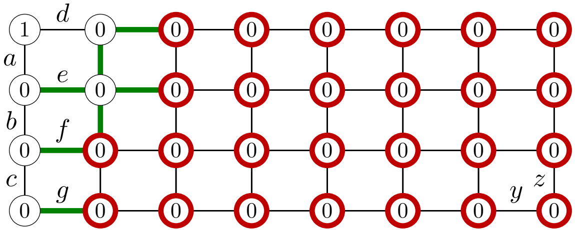

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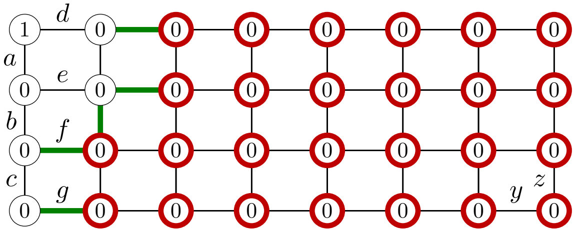

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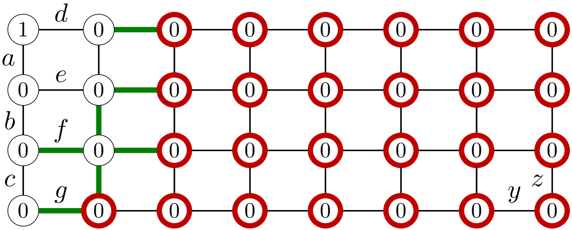

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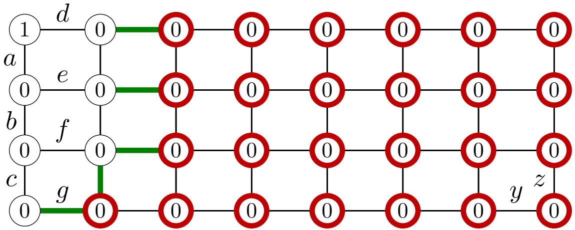

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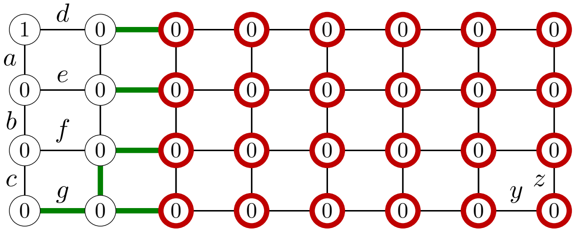

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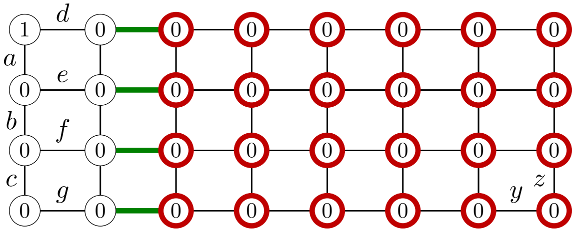

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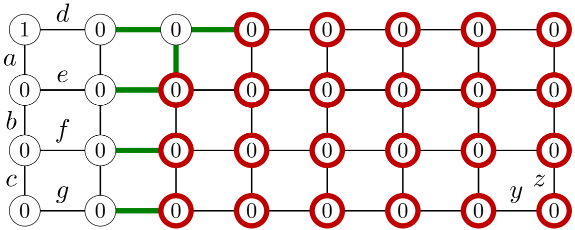

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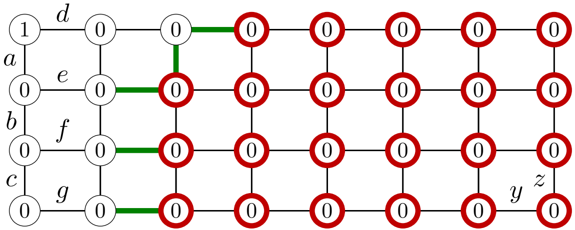

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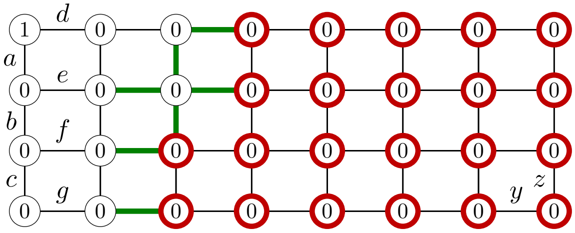

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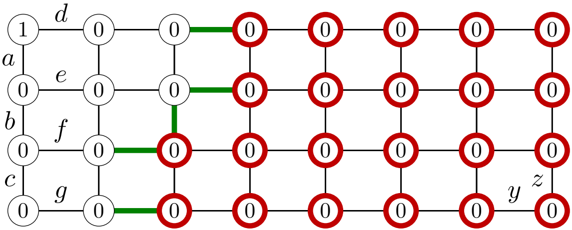

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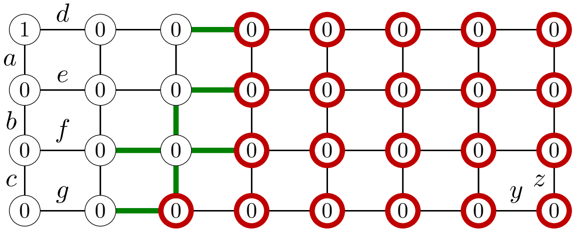

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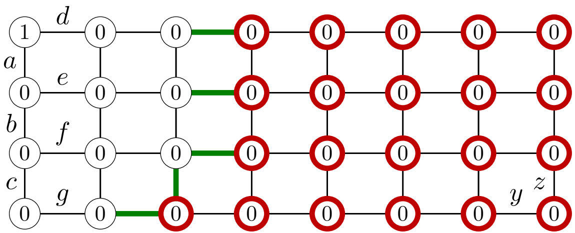

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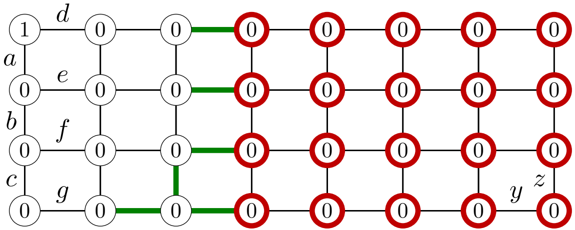

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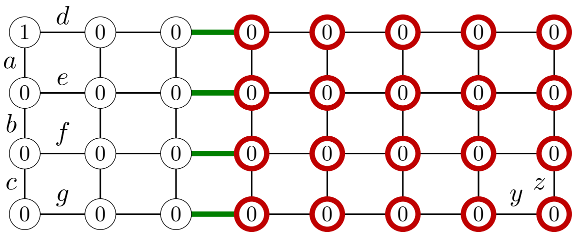

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$$c + g = 0$$
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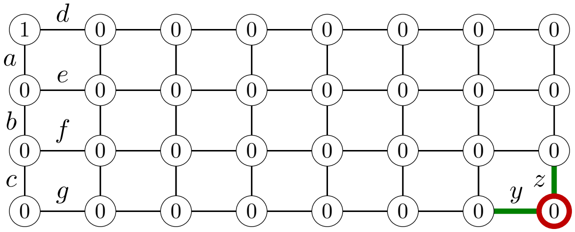
$$\vdots$$
$$y + z = 1$$



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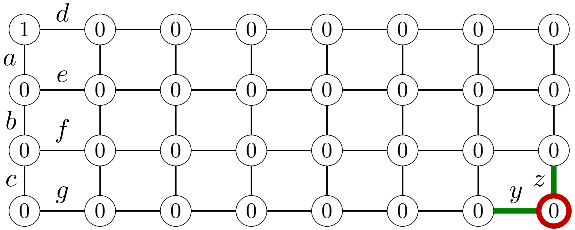

Small-Size "Dynamic programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses
- Total of mw summations

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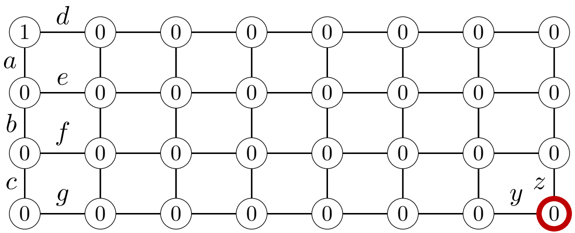
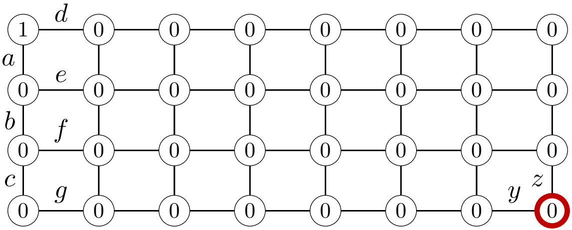
$$0 = 1$$



Small-Size "Dynamic programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow$ 2^w clauses
- Total of mw summations
- Small proof size $\mathcal{O}(mw2^w) = poly(m)$
  However, space $\approx$ size — superlinear!



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Polynomial Calculus

- Translate CNF to polynomials and do algebraic manipulations (so-called Gröbner basis computations) [CEI96, ABRW02]
- Much stronger proof system, but most results we covered for resolution carry over (proofs are different and harder, though!)
- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus
- Is it harder to build good algebraic SAT solvers, or is it just that too little work has been done (or both)?
- Some shortcut seems to be needed — full Gröbner basis computation does too much work

Cutting Planes

- Translate CNF to linear inequalities over the reals and prove no integral points in polytope [CCT87]
- Again much stronger proof system than resolution
- Very little known about size, space, or trade-offs
- Some work on pseudo-Boolean SAT solvers using (subset of) cutting planes
- Seems hard to make competitive with CDCL on CNFs — one key problem is to recover cardinality constraints
- But if cardinality constraints can be detected, solvers can do really well (at least on combinatorial benchmarks) [BBLM14]

Summing up

Overview of results on size-space trade-offs in proof complexity (more details in survey [Nor13])

- Resolution fairly well understood
- Most results in this talk carry over to polynomial calculus (algebraic reasoning)
- Cutting planes (geometric/pseudo-Boolean methods) very much less understood — several longstanding open questions

Summing up

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Two open questions:

- Do these time-space trade-offs actually show up in SAT solving practice? (Under investigation. . .)
- How can we build efficient SAT solvers based on stronger proof systems than resolution?

Summing up

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Thank you for your attention!

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