Leveraging Computational Complexity Theory for Verifiably Correct Combinatorial Optimization

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Computational Hardness in Theory

- Trained as computational complexity theorist
- Focus on problems in NP
- Prove unconditional lower bounds for bounded computational models
- Captures algorithmic approaches actually used in practice
- Except. . .

. . . And in Practice

- Combinatorial solving and optimization deals with NP-hard problems
- Show up all over the place, e.g.:
  - airline scheduling
  - logistics
  - hardware verification
  - donor-recipients matching for kidney transplants [MO12, BvdKM^{+}21]
- Lots of effort last decades into developing sophisticated so-called combinatorial solvers that often work amazingly well in practice!
  - Boolean satisfiability (SAT) solving [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
- Problems with cognitive dissonance — concepts like "strong exponential time hypothesis" just don't seem too relevant. . .
- Can computational complexity contribute anything?

The Dirty Little Secret...

- Solvers very fast, but ***sometimes wrong*** (even best commercial ones) [BLB10, CKSW13, AGJ$^+$18, GSD19, GS19]
- Even worse: No way of knowing for sure when errors happen
- Checking that a solution is feasible should be straightforward (though some solvers get even this wrong)
- But how to check the absence of solutions?
- Or that a solution is optimal?

What can be done about this?

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- **Formal verification**
  Prove that solver implementation adheres to formal specification
  Current techniques cannot scale to this level of complexity
- **Proof logging**
  Make solver certifying [ABM$^+$11, MMNS11] by outputting
  1. not only solution but also
  2. simple, machine-verifiable proof that solution is correct

Proof Logging with Certifying Solvers

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- be very simple (to increase trust)
- be powerful (to allow proof logging with minimal overhead)
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Computational complexity problems, but with a constructive angle!

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Success story for basic SAT solving: DRAT proof logging [HHW13a, HHW13b, WHH14]

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But has remained out of reach for stronger paradigms
And even for some advanced SAT solving techniques

Modern SAT Solving: Backtracking with a Twist

Try to build satisfying assignment — learn from mistakes

$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$

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Decision

Free choice to assign value to variable

Notation $w \stackrel{\mathsf{d}}{=} 0$

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Unit propagation

Forced choice to avoid falsifying clause

Given $w = 0$, clause $\overline{u} \vee w$ forces $u = 0$

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$y \stackrel{u \lor x \lor y}{=} 1$

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Conflict-Driven Clause Learning [MS96, BS97, MMZ+01]

Time to analyse this conflict!

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land \color{red}{(\overline{y} \lor \overline{z})} \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

$$\color{blue}{w \stackrel{\mathsf{d}}{=} 0}$$

$$u \stackrel{\overline{u} \lor w}{=} 0$$

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$$y \stackrel{u \lor x \lor y}{=} 1$$

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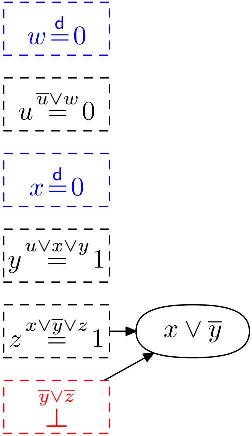

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Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
- $\overline{y} \vee \overline{z}$ wants $z = 0$
- Merge & remove z — must satisfy $x \vee \overline{y}$

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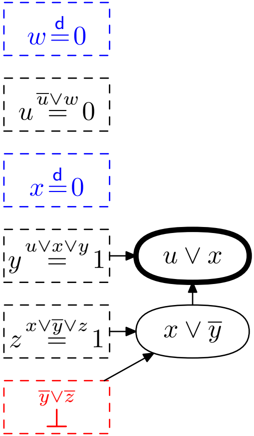

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Repeat until only 1 variable after last decision — learn that clause (1UIP) and backjump

Complete Example of CDCL Execution

Backjump: roll back max #decisions so that last variable still flips

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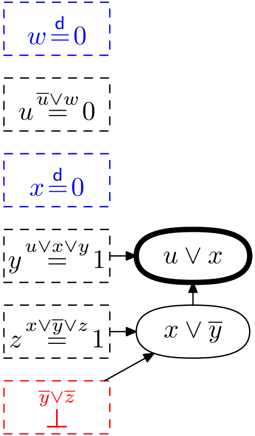

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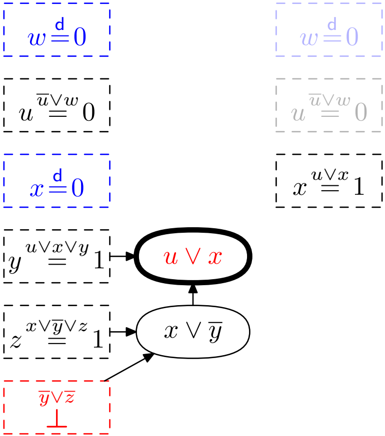

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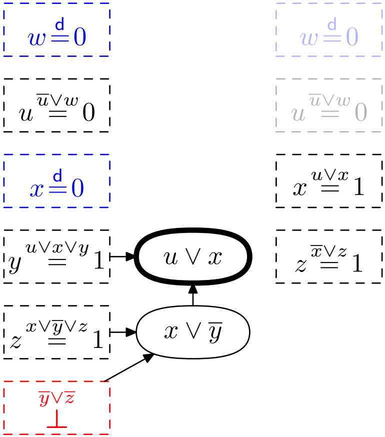

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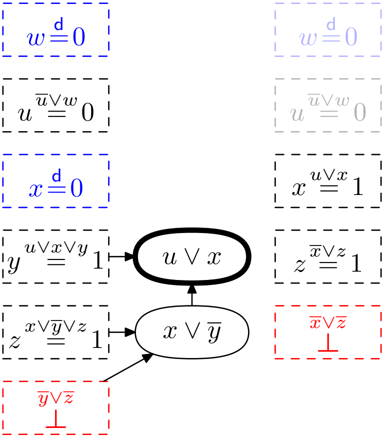

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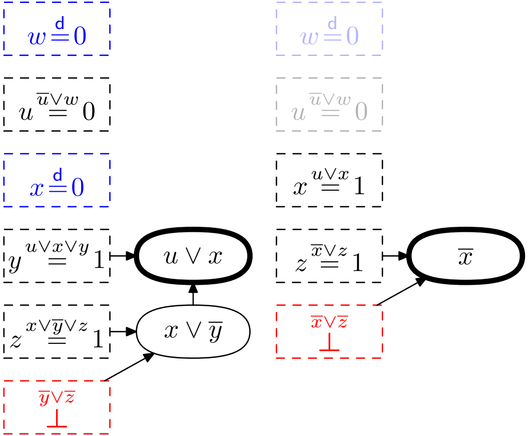

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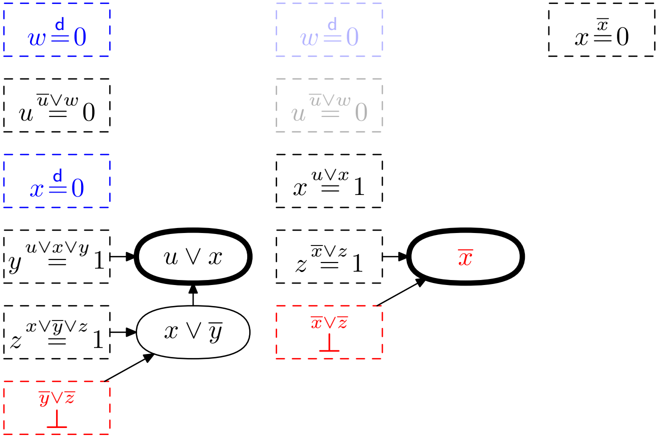

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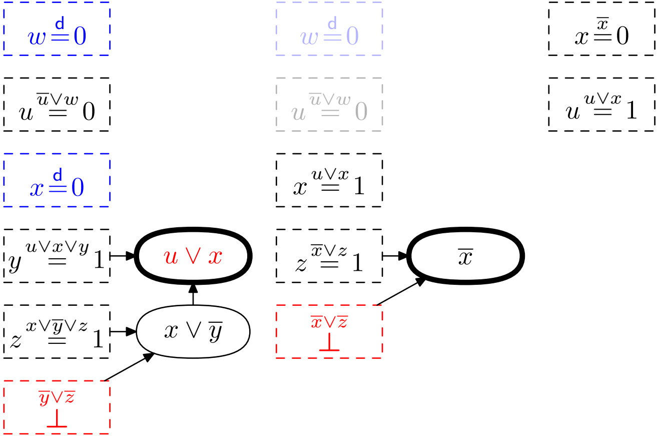

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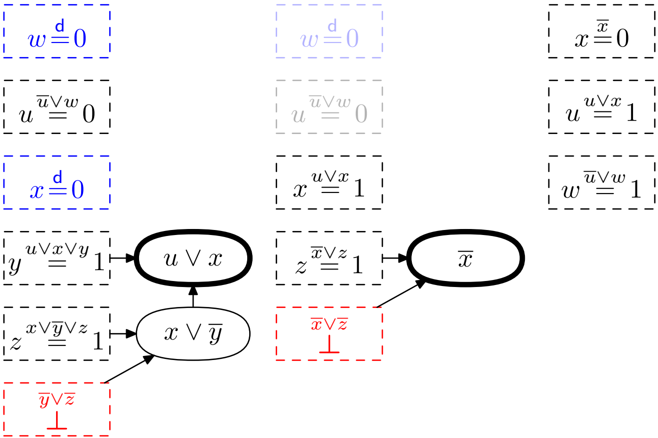

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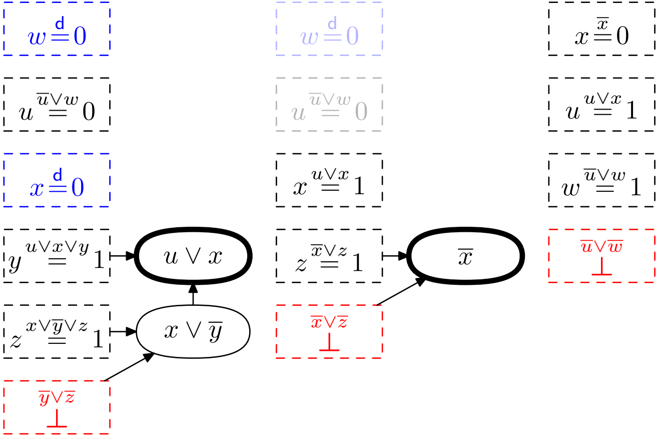

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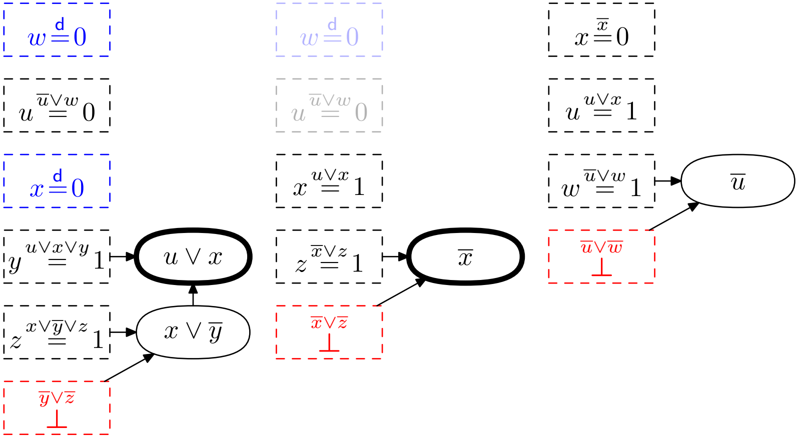

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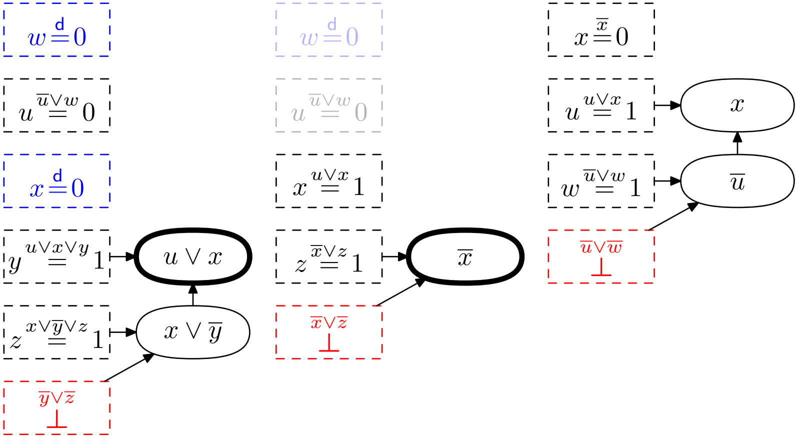

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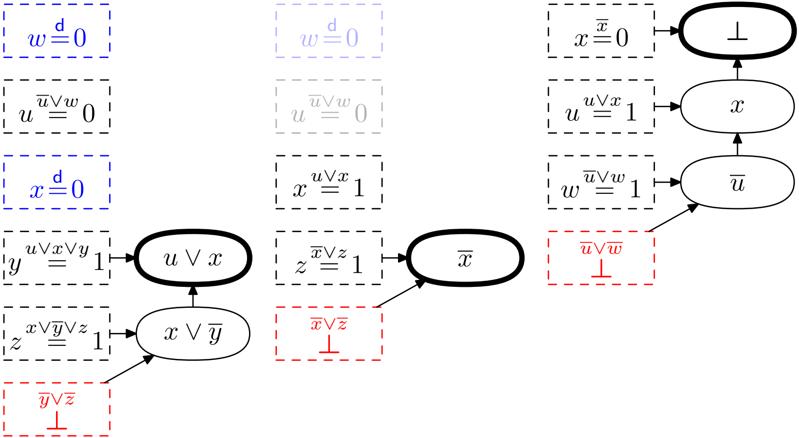

CDCL Reasoning and the Resolution Proof System

For CDCL proof logging, need proof system for unsatisfiable formulas
Focus on underlying method of reasoning

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Resolution proof system [Bla37, Rob65]

- Start with clauses of formula
- Derive new clauses by **resolution rule**

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

- Done when contradiction $\bot$ in form of empty clause derived

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When run on unsatisfiable formula, CDCL generates resolution proof*
(So lower bounds on proof size $\Rightarrow$ lower bounds on running time)

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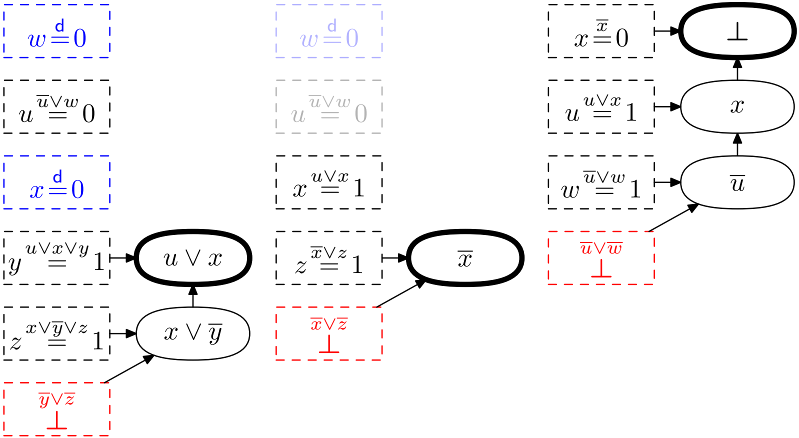
(*) Ignores pre- and inprocessing, but we will get there...

Resolution Proofs from CDCL Executions

Obtain resolution proof. . .

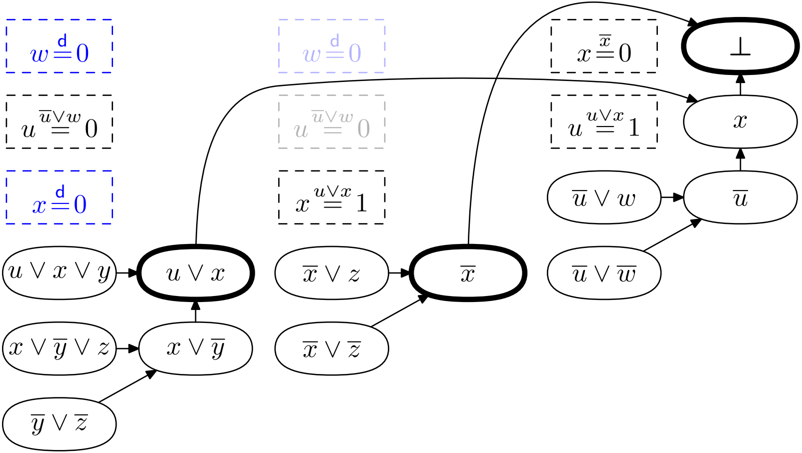
Resolution Proofs from CDCL Executions

Obtain resolution proof from our example CDCL execution...



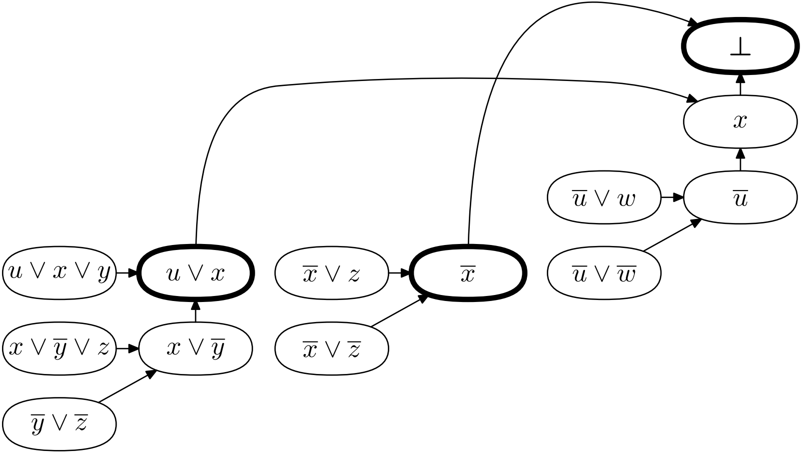
Resolution Proofs from CDCL Executions

Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



Resolution Proofs from CDCL Executions

Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



Reverse Unit Propagation

Reverse unit propagation (RUP) clause [GN03, Van08]

C is a RUP clause with respect to F if

- assigning C to false
- unit propagating on F until saturation
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Fact

All clauses learned by CDCL solver are RUP clauses

RUP Proofs

So shorter proof of unsatisfiability for

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

is sequence of RUP clauses

1. $u \lor x$
2. $\overline{x}$
3. $\bot$

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Extension Variables and Redundant Clauses

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Extended resolution proof system [Tse68, CR79] extremely powerful

Substitution Redundancy

- C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable
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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$
- Implication should be efficiently verifiable (e.g., all clauses in $(F \wedge C)\restriction_{\omega}$ should be RUP, say)

Next Challenge: Cardinality Constraints

Given clauses

$$x_1 \lor x_2 \lor x_3$$
$$x_1 \lor x_2 \lor x_4$$
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can deduce that

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Reasoning with cardinality constraints can solve pigeonhole principle efficiently, which is exponentially hard for basic CDCL [Hak85, BKS04]

Implemented in solver LINGELING [Lin], but no DRAT proof logging
Extended resolution can do it in theory, but efficiently in practice?!

Pseudo-Boolean Constraints

Pseudo-Boolean constraints are 0-1 integer linear constraints

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
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Some types of pseudo-Boolean constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging

Cutting planes proof system [CCT87]

Literal axioms $\dfrac{}{\ell_i \geq 0}$

Linear combination $\dfrac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B}$ $[c_A, c_B \geq 0]$

Division $\dfrac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil}$ $[c > 0]$

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Yields VeriPB proof system [EGMN20, GMN20, GMM$^+$20, GN21]

Recovering cardinality constraints from CNF

Clauses

$$x_1 \lor x_2 \lor x_3$$
$$x_1 \lor x_2 \lor x_4$$
$$x_1 \lor x_3 \lor x_4$$
$$x_2 \lor x_3 \lor x_4$$

Pseudo-Boolean constraints

$$x_1 + x_2 + x_3 \geq 1$$
$$x_1 + x_2 + x_4 \geq 1$$
$$x_1 + x_3 + x_4 \geq 1$$
$$x_2 + x_3 + x_4 \geq 1$$

Add all up

$$3x_1 + 3x_2 + 3x_3 + 3x_4 \geq 4$$

and divide by 3 to get

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

CDCL Solvers on Pseudo-Boolean Inputs

Can re-encode to CNF and run CDCL:

- MiniSat+ [ES06]
- Open-WBO [MML14]
- NaPS [SN15]

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to clauses with extension variables

$$s_{i,k} \Leftrightarrow \sum_{j=1}^{i} x_j \geq k$$

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$\overline{s}_{2,2} \vee s_{1,1}$

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$\overline{s}_{3,1} \vee s_{2,1} \vee x_3$

$\overline{s}_{3,2} \vee s_{2,1}$

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$s_{4,2}$

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$s_{4,2}$

How to know translation correct?
VeriPB can certify pseudo-Boolean-to-CNF rewriting [GMN21]

XOR Reasoning

Given clauses

$$x \lor y \lor z$$
$$x \lor \overline{y} \lor \overline{z}$$
$$\overline{x} \lor y \lor \overline{z}$$
$$\overline{x} \lor \overline{y} \lor z$$

and

$$y \lor z \lor w$$
$$y \lor \overline{z} \lor \overline{w}$$
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want to derive

$$x \lor \overline{w}$$
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This is just XOR reasoning:

$$x + y + z = 1 \pmod 2$$
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imply

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Could add XORs to language, but prefer to keep things super-simple and verifiable...

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Use substitution redundancy and fresh variables a,b to derive

$$x + y + z + 2a = 3$$
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("$=$" syntactic sugar for "$\geq$" plus "$\leq$")

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VeriPB can certify XOR reasoning [GN21]

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Symmetries

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- show up also in hard SAT benchmarks

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  1. Special-purpose, specific approach
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Interesting challenges for proof logging!

Challenges Beyond SAT

Proof logging for combinatorial optimization

- Maximum satisfiability (MaxSAT) solving
- Pseudo-Boolean optimization
- Mixed integer linear programming (some work in [CGS17, EG21])
- Constraint programming (some work in [EGMN20, GMN20, GMM+20])

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And more...

- Lots of challenging problems and interesting ideas
- Lots of interesting applications of proof logging — enables rigorous analysis of combinatorial solvers
- This talk would (hopefully) sound quite different in a year or two

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like a promising approach
- Leads to interesting computational complexity problems with constructive twist
- Cutting planes reasoning with pseudo-Boolean constraints might hit a sweet spot between simplicity and expressibility

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Thank you for your attention!

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