Relating Proof Complexity Measures and Practical Hardness of SAT

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SAT Interactions
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Joint work with Matti Järvisalo, Arie Matsliah, and Stanislav Živný

Proof Complexity and SAT Solving

Proof complexity

- Satsifiability fundamental problem in theoretical computer science
- SAT proven NP-complete by Stephen Cook in 1971
- Hence totally intractable in worst case (probably)
- One of the million dollar "Millennium Problems"

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- But best solvers still based on methods from early 1960s
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What (if anything) can proof complexity say about this?

Outline

1. SAT solving and Proof Complexity
   - SAT solving and DPLL
   - Proof Complexity and Resolution
   - Our Results
2. Experiments
   - Benchmark Formulas
   - Set-up
   - Results
3. Directions for Future Research

From Proving Tautologies To Disproving CNF Formulas

Conjunctive normal form (CNF)

ANDs of ORs of variables or negated variables
(or conjunctions of disjunctive clauses)

Example:

$$
(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})
$$
$$
\land\ (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})
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Proving that a formula in propositional logic is **always** satisfied

$\Updownarrow$

Proving that a CNF formula is **never** satisfied

Some Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (assume k fixed)
- Refer to clauses of CNF formula as axioms
  (as opposed to derived clauses)

The DPLL Method

Based on [Davis & Putnam '60] and [Davis, Logemann & Loveland '62]

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- Set $x = 0$, simplify F and try to refute recursively

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- Otherwise pick some variable x in F
- Set $x = 0$, simplify F and try to refute recursively
- Set $x = 1$, simplify F and try to refute recursively
- If result in both cases "unsatisfiable", then report "unsatisfiable"

A DPLL Toy Example

$$
\begin{aligned}
F = \quad & (x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u}) \\
& \wedge (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w})
\end{aligned}
$$

A DPLL Toy Example

$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when falsfied clause found

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$$F = (x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u}) \wedge (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w})$$

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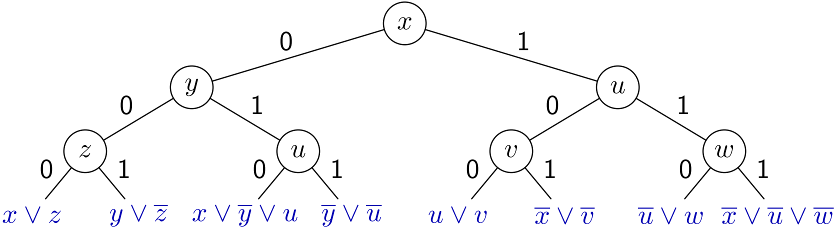

A DPLL Toy Example

$$F = (\quad z) \wedge (y \vee \overline{z}) \wedge (\quad \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u})$$
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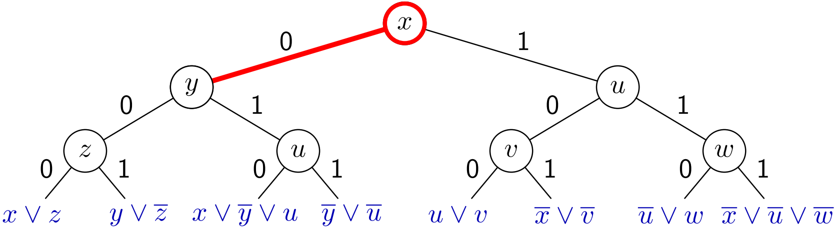

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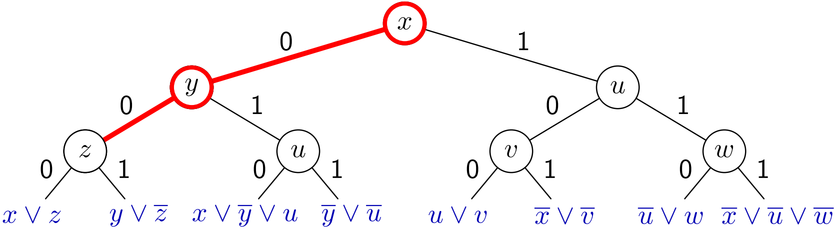

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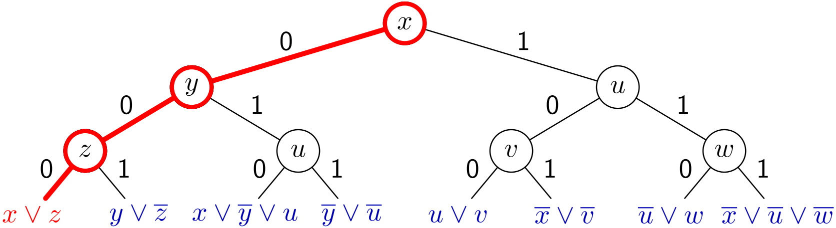

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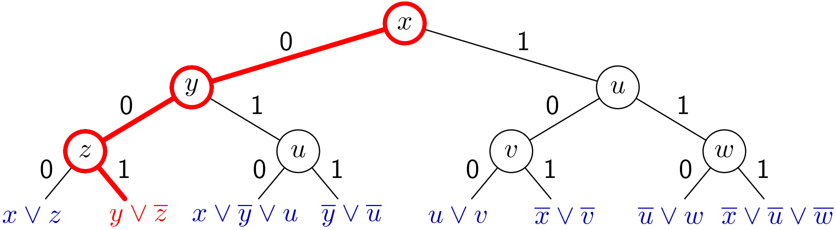

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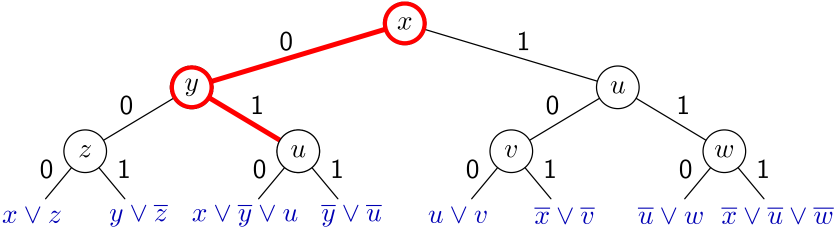

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$$F = (\qquad z) \land (y \lor \overline{z}) \land (\qquad\qquad\quad) \land (\qquad \overline{u}) \\ \land (\qquad v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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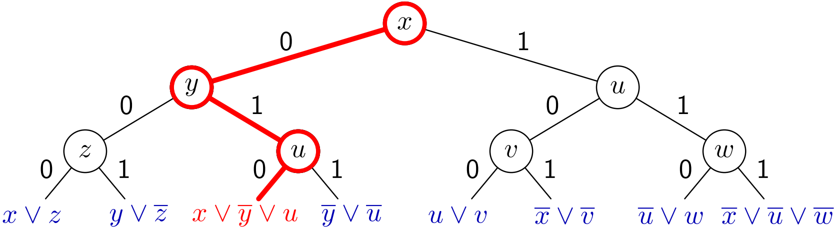

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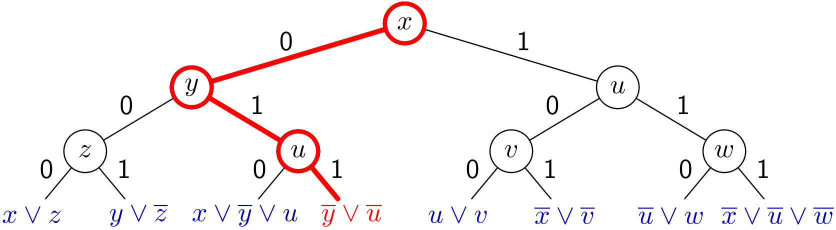

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$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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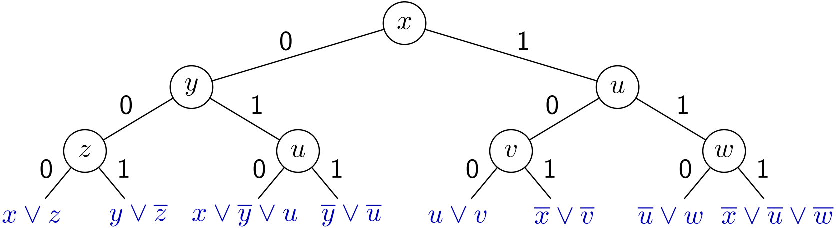

State-of-the-art DPLL SAT solvers

Many more ingredients in modern SAT solvers, for instance:

- Choice of pivot variables crucial
- When reaching falsified clause, compute why partial assignment failed — add this info to formula as new clause
  Conflict-driven clause learning (CDCL)
- Every once in a while, restart (but save computed info)

Proof Complexity

Proof search algorithm: defines proof system with derivation rules

Proof complexity: study of proofs in such systems

- Lower bounds: no algorithm can do better (even optimal one always guessing the right move)
- Upper bounds: gives hope for good algorithms if we can search for proofs in system efficiently

Resolution

Resolution rule:

$$\frac{B \vee x \quad C \vee \overline{x}}{B \vee C}$$

Resolution

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$$\frac{B \lor x \quad C \lor \overline{x}}{B \lor C}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $C_1, C_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

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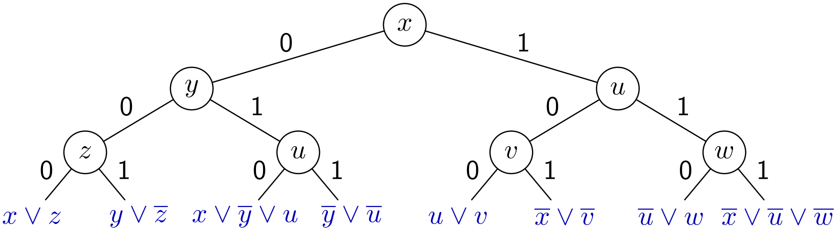
Observation

If F is a satisfiable CNF formula and D is derived from clauses $C_1, C_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

Prove F unsatisfiable by deriving the unsatisfiable empty clause $\bot$ from F by resolution

CDCL Solvers Generate Resolution Proofs

Simple example for DPLL:



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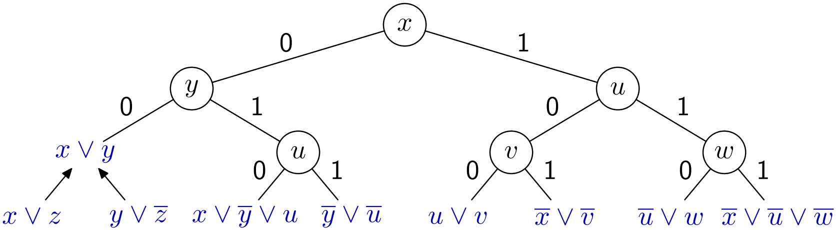

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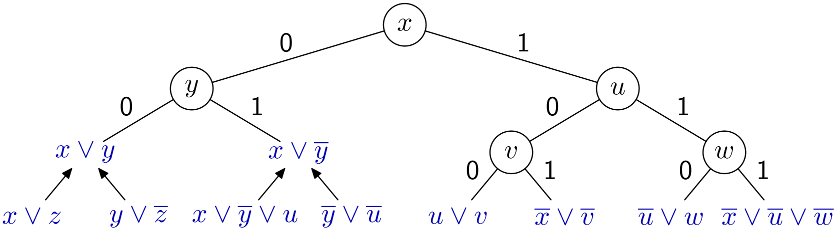

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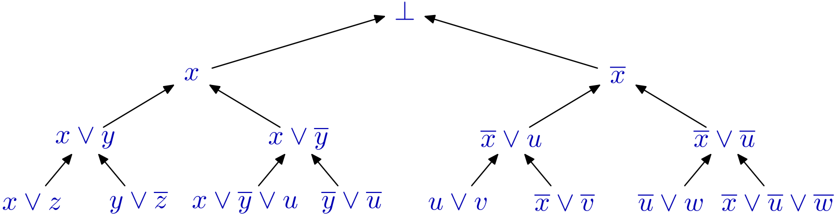

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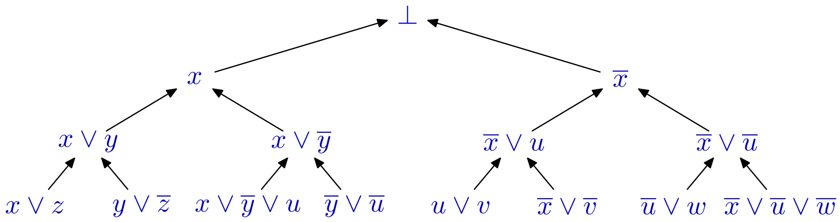


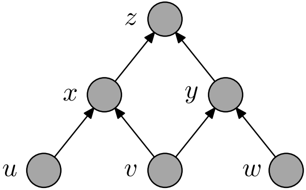
- Conflict-driven clause learning adds "shortcut edges" in tree
- But still yields resolution proof
- True also for (most) preprocessing techniques

The Theoretical Model

- Goal: Refute given CNF formula (i.e., prove it is unsatisfiable)
- Proof system operates with disjunctive clauses
- Proof/refutation is "presented on blackboard"
- Derivation steps:
  - Write down clauses of CNF formula being refuted (axiom clauses)
  - Infer new clauses by resolution rule
  - Erase clauses that are not currently needed (to save space on blackboard)
- Refutation ends when empty clause $\bot$ is derived

Example CNF Formula

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

Example CNF Formula

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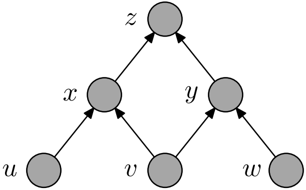

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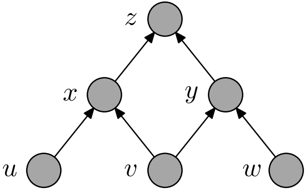

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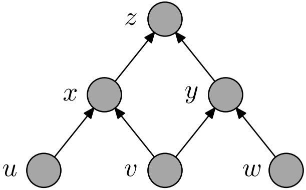

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Example Resolution Refutation

1. u
2. v
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4. $\overline{u} \lor \overline{v} \lor x$
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6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	0
largest clause seen on board	0
max # lines on board	0

Can write down axioms,
erase used clauses or
infer new clauses by resolution rule
(but only from clauses currently on the board!)

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	1
largest clause seen on board	1
max # lines on board	1

u

Write down axiom 1: u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	2
largest clause seen on board	1
max # lines on board	2

u

v

Write down axiom 1: u

Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
largest clause seen on board	3
max # lines on board	3

u
v
$\overline{u} \vee \overline{v} \vee x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
largest clause seen on board	3
max # lines on board	3

u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
largest clause seen on board	3
max # lines on board	4

u
v
$\overline{u} \vee \overline{v} \vee x$
$\overline{v} \vee x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
largest clause seen on board	3
max # lines on board	4

u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
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4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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Blackboard bookkeeping	
total # clauses on board	4
largest clause seen on board	3
max # lines on board	4

u
v
$\overline{v} \lor x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
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largest clause seen on board	3
max # lines on board	4

u
v
$\overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$
Erase the clause u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
largest clause seen on board	3
max # lines on board	4

v
$\overline{v} \vee x$

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$
Erase the clause $\overline{u} \vee \overline{v} \vee x$
Erase the clause u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
largest clause seen on board	3
max # lines on board	4

v

$\overline{v} \lor x$

u and $\overline{u} \lor \overline{v} \lor x$

Erase the clause $\overline{u} \lor \overline{v} \lor x$

Erase the clause u

Infer x from

v and $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
largest clause seen on board	3
max # lines on board	4

v

$\overline{v} \vee x$

x

u and $\overline{u} \vee \overline{v} \vee x$

Erase the clause $\overline{u} \vee \overline{v} \vee x$

Erase the clause u

Infer x from

v and $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
largest clause seen on board	3
max # lines on board	4

v
$\overline{v} \vee x$
x

Erase the clause $\overline{u} \vee \overline{v} \vee x$
Erase the clause u
Infer x from
v and $\overline{v} \vee x$
Erase the clause $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
largest clause seen on board	3
max # lines on board	4

v
x

Erase the clause $\overline{u} \vee \overline{v} \vee x$
Erase the clause u
Infer x from
v and $\overline{v} \vee x$
Erase the clause $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
largest clause seen on board	3
max # lines on board	4

v
x

Erase the clause u
Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$
Erase the clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
largest clause seen on board	3
max # lines on board	4

x

Erase the clause u
Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$
Erase the clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
largest clause seen on board	3
max # lines on board	4

Board:

x

$\overline{x} \lor \overline{y} \lor z$

Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$
Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
largest clause seen on board	3
max # lines on board	4

Board:

x

$\overline{x} \lor \overline{y} \lor z$

Erase the clause $\overline{v} \lor x$

Erase the clause v

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Infer $\overline{y} \lor z$ from

x and $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
largest clause seen on board	3
max # lines on board	4

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the clause $\overline{v} \lor x$
Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
largest clause seen on board	3
max # lines on board	4

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
largest clause seen on board	3
max # lines on board	4

x
$\overline{y} \lor z$

Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
largest clause seen on board	3
max # lines on board	4

x

$\overline{y} \lor z$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Infer $\overline{y} \lor z$ from

x and $\overline{x} \lor \overline{y} \lor z$

Erase the clause $\overline{x} \lor \overline{y} \lor z$

Erase the clause x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
largest clause seen on board	3
max # lines on board	4

$\overline{y} \lor z$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
  x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$
Erase the clause x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
largest clause seen on board	3
max # lines on board	4

$\overline{y} \vee z$

$\overline{v} \vee \overline{w} \vee y$

Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the clause $\overline{x} \vee \overline{y} \vee z$
Erase the clause x
Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
largest clause seen on board	3
max # lines on board	4

$\overline{y} \lor z$

$\overline{v} \lor \overline{w} \lor y$

Erase the clause $\overline{x} \lor \overline{y} \lor z$

Erase the clause x

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
largest clause seen on board	3
max # lines on board	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the clause $\overline{x} \lor \overline{y} \lor z$
Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
largest clause seen on board	3
max # lines on board	4

$\overline{y} \lor z$

$\overline{v} \lor \overline{w} \lor y$

$\overline{v} \lor \overline{w} \lor z$

Erase the clause x

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Erase the clause $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
largest clause seen on board	3
max # lines on board	4

$\overline{y} \lor z$

$\overline{v} \lor \overline{w} \lor z$

Erase the clause x

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Erase the clause $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
largest clause seen on board	3
max # lines on board	4

$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee z$

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
largest clause seen on board	3
max # lines on board	4

$\overline{v} \vee \overline{w} \vee z$

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	10
largest clause seen on board	3
max # lines on board	4

$\overline{v} \lor \overline{w} \lor z$

v

Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$
Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	11
largest clause seen on board	3
max # lines on board	4

$\overline{v} \lor \overline{w} \lor z$

v

w

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Erase the clause $\overline{v} \lor \overline{w} \lor y$

Erase the clause $\overline{y} \lor z$

Write down axiom 2: v

Write down axiom 3: w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
largest clause seen on board	3
max # lines on board	4

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$
Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
largest clause seen on board	3
max # lines on board	4

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
largest clause seen on board	3
max # lines on board	5

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
largest clause seen on board	3
max # lines on board	5

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
largest clause seen on board	3
max # lines on board	5

$\overline{v} \lor \overline{w} \lor z$

w

$\overline{z}$

$\overline{w} \lor z$

Write down axiom 3: w

Write down axiom 7: $\overline{z}$

Infer $\overline{w} \lor z$ from

v and $\overline{v} \lor \overline{w} \lor z$

Erase the clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
largest clause seen on board	3
max # lines on board	5

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
largest clause seen on board	3
max # lines on board	5

w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
largest clause seen on board	3
max # lines on board	5

w
$\overline{z}$
$\overline{w} \lor z$

v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
largest clause seen on board	3
max # lines on board	5

w
$\overline{z}$
$\overline{w} \lor z$
z

v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
largest clause seen on board	3
max # lines on board	5

w
$\overline{z}$
$\overline{w} \vee z$
z

Erase the clause v
Erase the clause $\overline{v} \vee \overline{w} \vee z$
Infer z from
w and $\overline{w} \vee z$
Erase the clause w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
largest clause seen on board	3
max # lines on board	5

$\overline{z}$
$\overline{w} \lor z$
z

Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the clause w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
largest clause seen on board	3
max # lines on board	5

$\overline{z}$
$\overline{w} \lor z$
z

Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the clause w
Erase the clause $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
largest clause seen on board	3
max # lines on board	5

$\overline{z}$
z

Erase the clause $\overline{v} \vee \overline{w} \vee z$

Infer z from

w and $\overline{w} \vee z$

Erase the clause w

Erase the clause $\overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
largest clause seen on board	3
max # lines on board	5

$\overline{z}$
z

w and $\overline{w} \vee z$
Erase the clause w
Erase the clause $\overline{w} \vee z$
Infer $\perp$ from
$\overline{z}$ and z

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	15
largest clause seen on board	3
max # lines on board	5

$\overline{z}$
z
$\bot$

w and $\overline{w} \lor z$
Erase the clause w
Erase the clause $\overline{w} \lor z$
Infer $\bot$ from
$\overline{z}$ and z

Complexity Measures for Resolution

Let n = size of formula

Length

clauses in refutation — at most $\exp(n)$ [in our example: 15]

Width

Size of largest clause in refutation — at most n [in our example: 3]

Space

Max # clauses one needs to remember when "verifying correctness of refutation on blackboard" — at most n (!) [in our example: 5]

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This work can be viewed as implementing program outlined in [ABLM08]

Result 1: Separation of Space and Tree-like Space

We don't believe in tree-like space as hardness measure

- Tree-like space tightly connected with tree-like length
- Corresponds to DPLL without clause learning
- Would suggest CDCL doesn't buy you anything

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We prove first asymptotic separation of space and tree-like space

Theorem

There are formulas requiring space $\mathcal{O}(1)$ for which tree-like space grows like $\Omega(\log n)$

Only constant-factor separation known before [Esteban & Torán '03]

Result 2: Small Backdoor Sets Imply Small Space

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We show connections between backdoors and space complexity (elaborating on [ABLM08])

Theorem (Informal)

If a formula has a small backdoor set, then it requires small space

Result 3: Hardness in Practice Correlates with Space

Recall

$$\log \text{length} \leq \text{width} \leq \text{space} \leq \text{tree-like space}$$

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- Is running time essentially the same?
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Experimental results

Running times seem to correlate with space complexity**

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(*) But such formulas are nontrivial to find
(**) With some caveats to be discussed later

How to Get Hold of Good Benchmark Formulas?

Questions about space complexity and time-space trade-offs fundamental in theoretical computer science

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In particular, well-studied (and well-understood) for pebble games modelling calculations described by DAGs ([Cook & Sethi '76] and others)

- Time needed for calculation: # pebbling moves
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- Time needed for calculation: # pebbling moves
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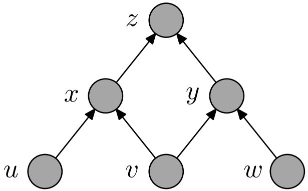
Some quick graph terminology

- DAGs consist of vertices with directed edges between them
- vertices with no incoming edges: sources
- vertices with no outgoing edges: sinks

The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



# moves	0
Current # pebbles	0
Max # pebbles so far	0

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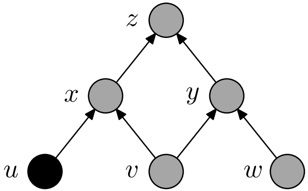

# moves	1
Current # pebbles	1
Max # pebbles so far	1

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them

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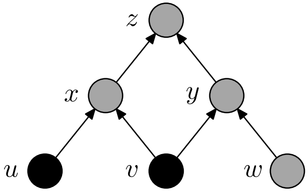

# moves	2
Current # pebbles	2
Max # pebbles so far	2

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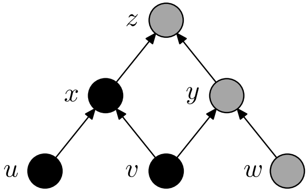

# moves	3
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them

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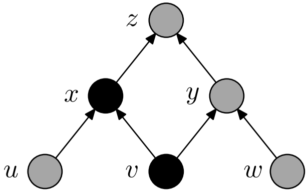

# moves	4
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex

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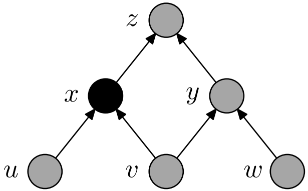

# moves	5
Current # pebbles	1
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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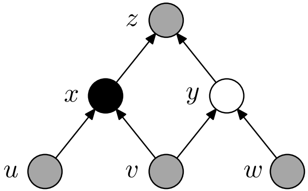

# moves	6
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
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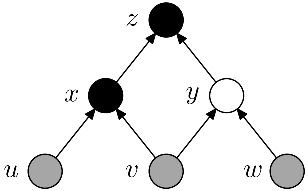

# moves	7
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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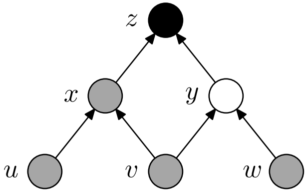

# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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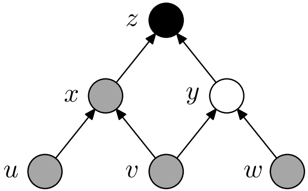

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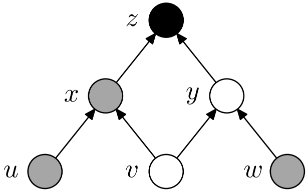

# moves	9
Current # pebbles	3
Max # pebbles so far	3

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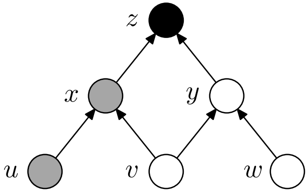

# moves	10
Current # pebbles	4
Max # pebbles so far	4

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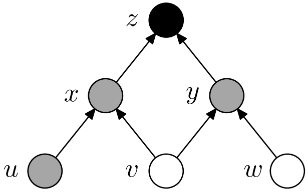

# moves	11
Current # pebbles	3
Max # pebbles so far	4

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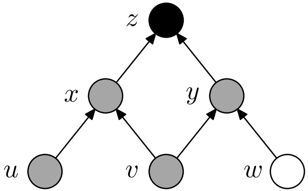

# moves	12
Current # pebbles	2
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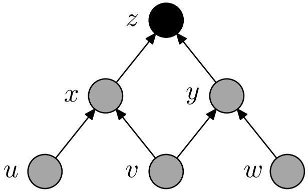

# moves	13
Current # pebbles	1
Max # pebbles so far	4

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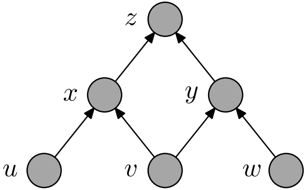
Use Pebbling Formulas...

CNF formulas encoding so-called pebble games on DAGs

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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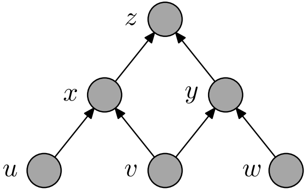

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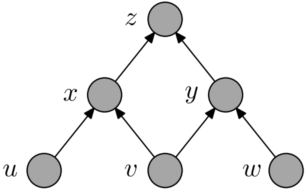

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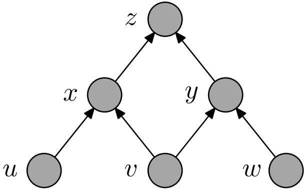

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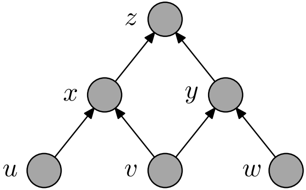

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Extensive literature on pebbling time-space trade-offs from 1970s and 80s

Pebbling formulas studied by [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and others

Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling formulas.

Use Pebbling Formulas. . .

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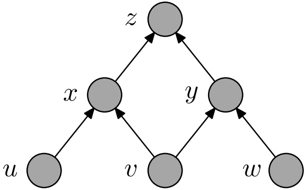

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. . . with Functions Substituted for Variables

Won't work — pebbling formulas solved by unit propagation, so supereasy

Make formula harder by substituting $x_1 \oplus x_2$ for every variable x (also works for other Boolean functions with "right" properties):

$$\overline{x} \vee y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$$

$$\Downarrow$$

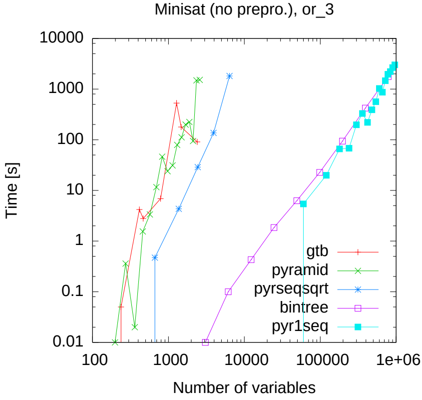
$$\begin{aligned} & (x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \\ \wedge\ & (x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee y_1 \vee y_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2) \end{aligned}$$

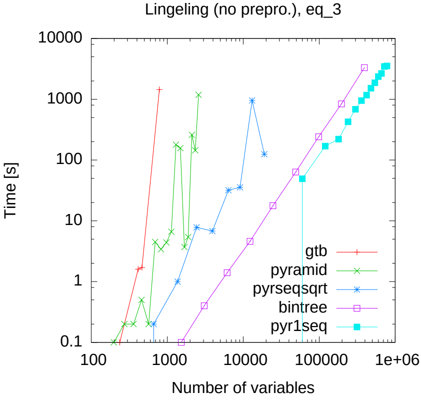
Now CNF formula inherits pebbling graph properties!

About the Experiments

- 12 graph families with varying space complexity
- 8 different substitution functions
- Total of 96 formula families with around 50 instances per family
- CDCL solvers Minisat 2.2.0 and Lingeling version 774
- Experiments
  - with and without preprocessing
  - with and without random shuffling of clauses and variables
- Intel Core i5-2500 3.3-GHz quad-core CPU with 8 GB of memory
- Time-out 1 hour per instance
- Massive amounts of data...

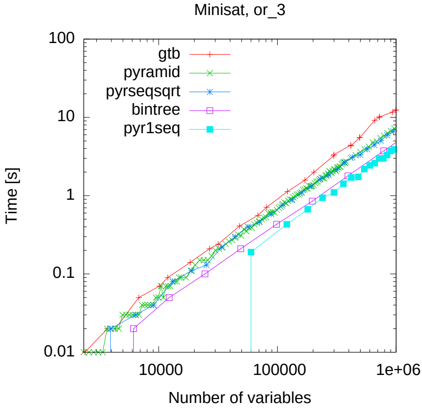
Example Results Without Preprocessing

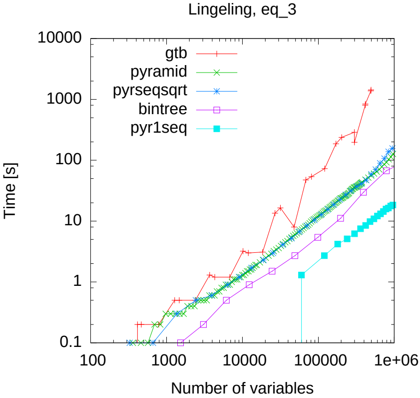




Looks nice. . . Easy formulas solved fast and hard formulas take longer time

Example Results with Preprocessing





Less nice. . . Which is not surprising

Caveats and Issues

Preprocessing dampens correlations

- To be expected — space of proof not captured during preprocessing
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Varying width and space independently would be more convincing

- Very true, but provably impossible since space $\geq$ width
- Want to see if space is "more fine-grained" hardness indicator

Some Open Questions

- Get similar results with preprocessing turned on?
- Do theoretical time-space trade-offs turn up in practice for CDCL solvers?
- How does space complexity (and other complexity measures) correlate with running time for algebraic SAT solvers?
- Understand relations of measures such as space and degree better for algebraic solvers (corresponding to polynomial calculus proof system)
- Build better SAT solvers based on algebra or geometry!

Summing up

- Modern CDCL SAT solvers amazingly successful in practice
- But poorly understood which formulas are easy or hard
- We propose space complexity as a measure of hardness in practice
- Don't claim conclusive evidence, but nontrivial correlations
- Believe there are more connections between proof complexity and SAT solving worth exploring

Thank you for your attention!