Using Pseudo-Width to Prove Lower Bounds for Highly Overconstrained Formulas

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KTH Royal Institute of Technology
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Computational Complexity of Discrete Problems
Schloss Dagstuhl – Leibniz Center for Informatics
March 22, 2019

Joint work with Susanna F. de Rezende, Kilian Risse, and Dmitry Sokolov

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Thanks for help with the slides!

Proof Complexity

- Study of efficiently verifiable certificates of unsatisfiability
- Example: Is the following CNF formula satisfiable?

$$(\overline{z} \vee y) \wedge (z \vee \overline{y} \vee \overline{x}) \wedge (z \vee y) \wedge (\overline{y} \vee x) \wedge (\overline{z} \vee \overline{x})$$

- Study the power of different methods of reasoning (a.k.a. proof systems) in propositional logic
- This talk: resolution

Motivation for Proof Complexity

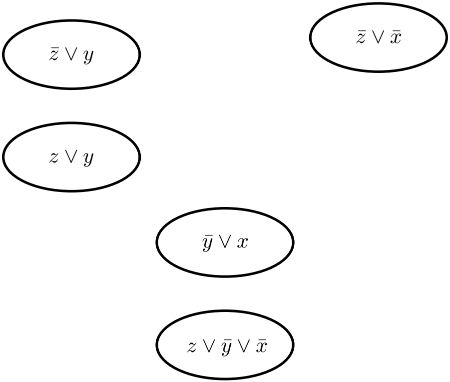
1. Separate NP and coNP
2. Understand how much reasoning power required to prove different mathematical statements
3. Analyse applied satisfiability algorithms (SAT solvers)

Just To Make Sure We're on the Same Page...

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- Empty clause (with no literals) denoted $\perp$ = (contradiction)

Resolution Proof System

- Derive new clauses using resolution rule: $\frac{C \lor x \quad D \lor \overline{x}}{C \lor D}$
- Certify unsatisfiability by deriving empty clause $\perp$
- Proof of unsatisfiability = refutation



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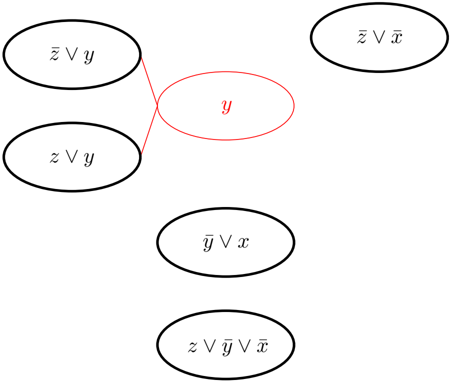

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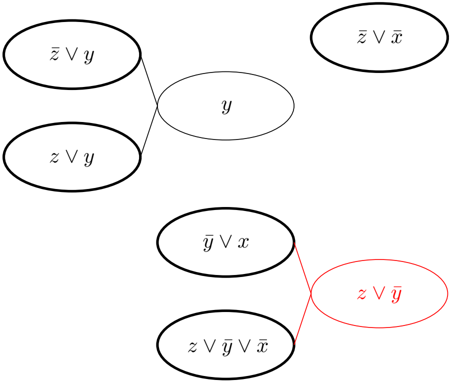

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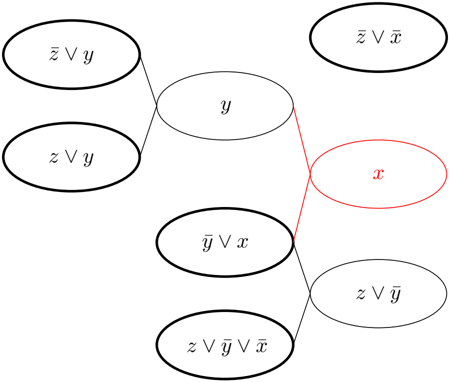

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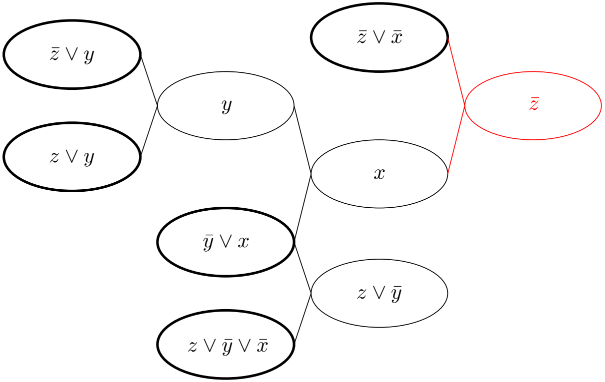

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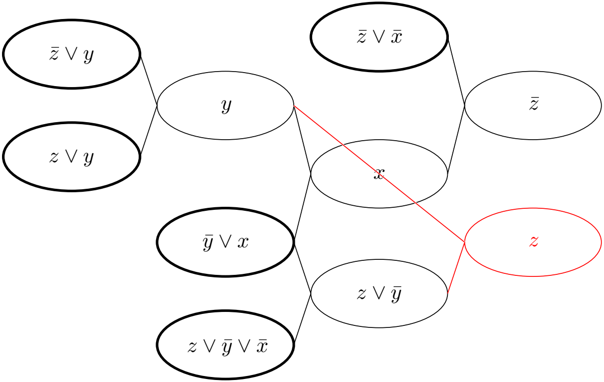

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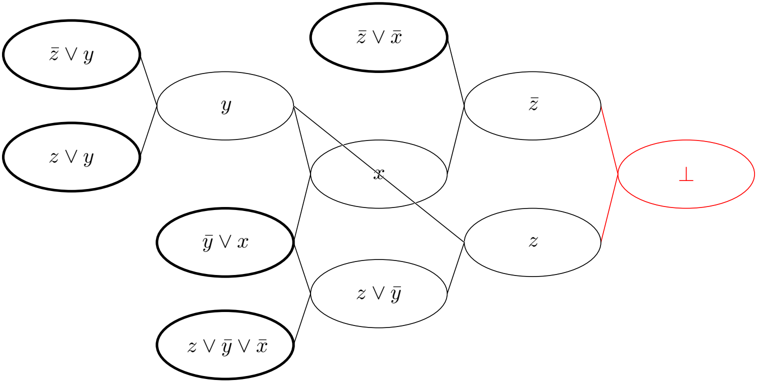
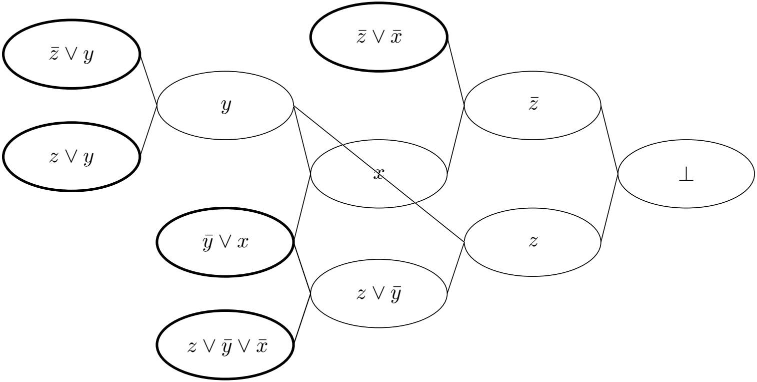


Complexity Measures for Resolution



- Length of refutation = #clauses (11 in our example)

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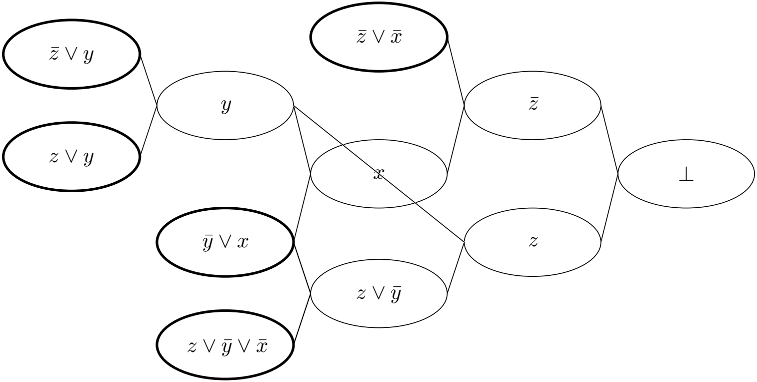

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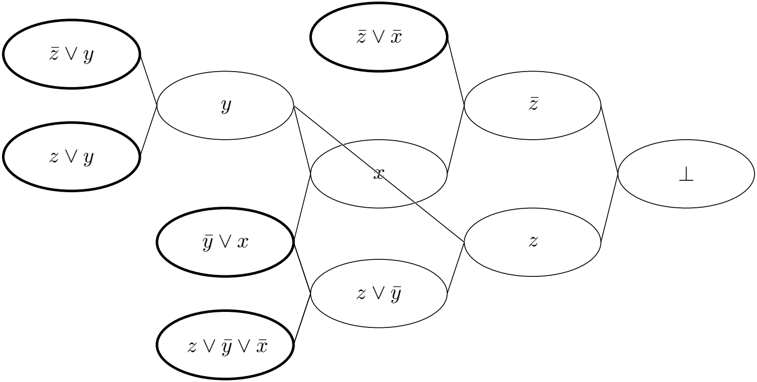

- Length of refutation = #clauses (11 in our example)
- Width of refutation = #literals in largest clause (3 in our example)
- Minimize over all refutations to define length $L(F \vdash \perp)$ and width $W(F \vdash \perp)$ of refuting formula F

Size-Width Lower Bound

Ben-Sasson & Wigderson [BW01]

$$L(F \vdash \bot) = \exp\left(\Omega\left(\frac{W(F \vdash \bot)^2}{\#\text{variables in } F}\right)\right)$$

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- Linear lower bounds on width $\Rightarrow$ exponential lower bounds on length

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- Linear lower bounds on width $\Rightarrow$ exponential lower bounds on length
- Can be used to prove almost all resolution lower bounds:
  - Pigeonhole principle formulas [Hak85]
  - Tseitin formulas [Urq87]
  - Random k-CNF formulas [CS88, BKPS02]
  - ...

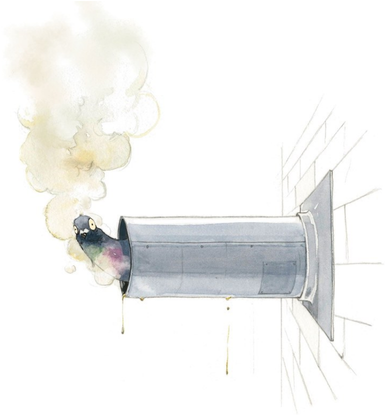
Open Problems

- So are we done with resolution? Not quite...
- Size-width lower bound yields nothing for width $\lessapprox \sqrt{\#\text{variables}}$
- This is essentially tight by [BG01]
- Interesting challenges for resolution lower bounds e.g.:
  - k-clique formulas
  - Pseudo-random generator formulas
  - Weak pigeonhole principle formulas (highly overconstrained)

This talk

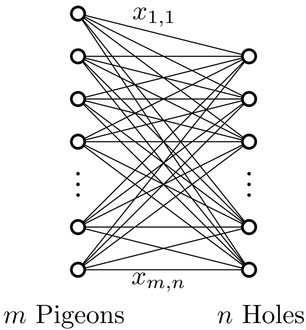
- Strong lower bounds for weak pigeonhole principle formulas
- Using and refining Razborov's pseudo-width method [Raz03, Raz04b]
- Seems like a very powerful tool that could be useful elsewhere

Pigeonhole Principle (PHP) Formulas



Pigeonhole Principle (PHP) Formulas

- One variable per edge:
  $x_{i,j}$ for $i \in [m]$ and $j \in [n]$
- Pigeon axioms: At least 1 hole
  $\bigvee_{j \in [n]} x_{i,j}$ (for $i \in [m]$)
- Hole axioms: At most 1 pigeon
  $\overline{x}_{i,j} \vee \overline{x}_{i',j}$ (for $i \neq i' \in [m], j \in [n]$)



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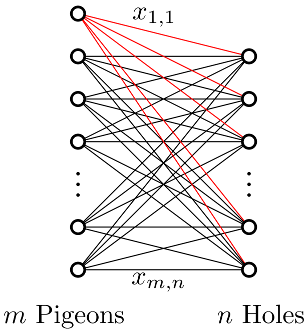

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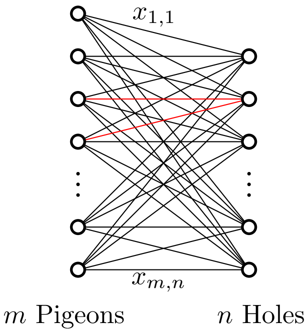

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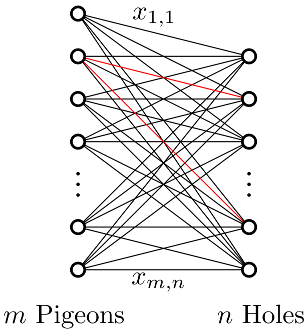

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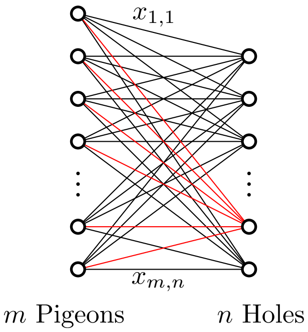
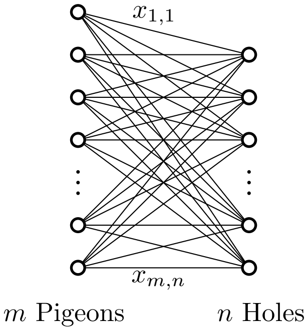


Pigeonhole Principle and Resolution: Some History

- Haken [Hak85]:
  $L(\mathit{Onto}\text{-}\mathit{FPHP}_n^{n+1} \vdash \perp) = \exp(\Omega(n))$



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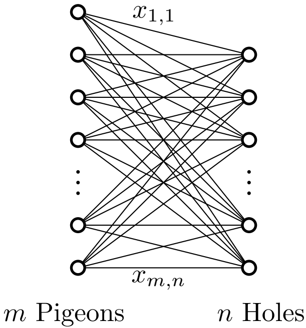

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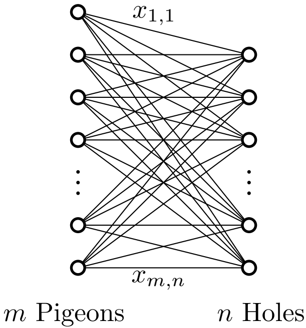

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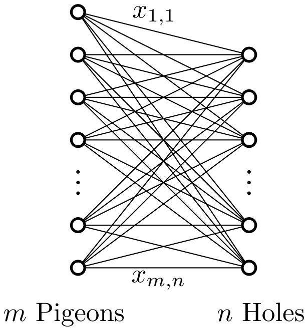

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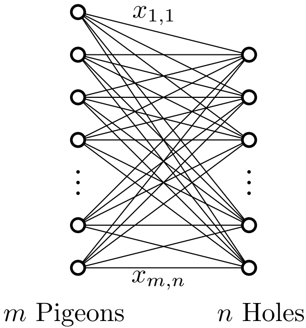

(Much more info in Razborov's survey on PHP in proof complexity [Raz02])

Pigeonhole Principle on Graphs

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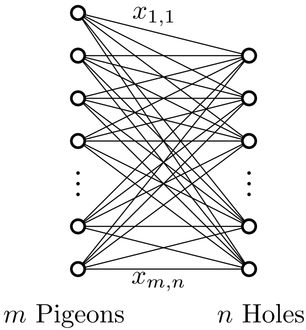
- Replace complete graph by "good" sparse graph, restricting pigeon choices



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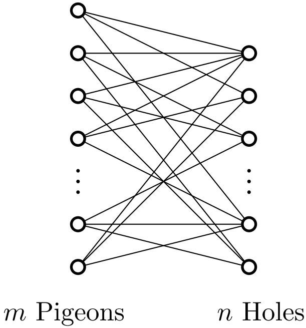

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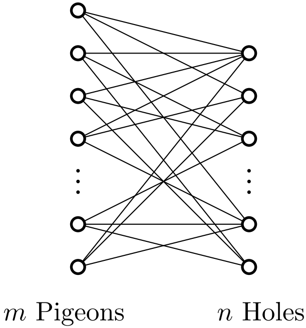
- Intuitively, graph is "good" if any small set of pigeons has many partial matchings



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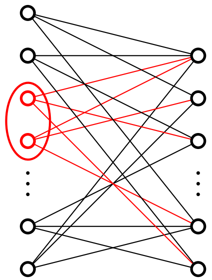

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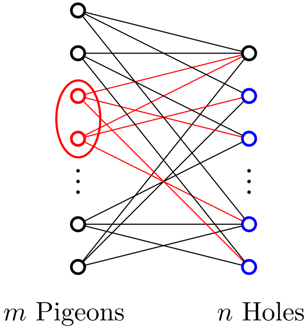
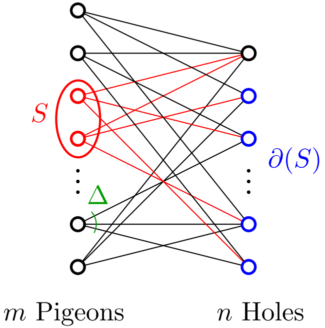


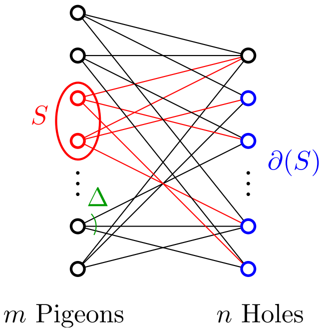
Pigeonhole Principle on Graphs

- Replace complete graph by "good" sparse graph, restricting pigeon choices
- Intuitively, graph is "good" if any small set of pigeons has many partial matchings
- (r, Δ, c)-boundary expander:
  1. every pigeon has degree $\leq \Delta$
  2. all sets $S \subseteq [m]$ of size $\leq r$ have $\geq c \cdot |S|$ unique neighbours



Lower Bounds for Graph PHP Formulas on Expanders

- Ben-Sasson & Wigderson [BW01]:
  For $r = \Omega(n/\log m)$, $\Delta = \log m$ and $c = \frac{3}{4}\log m$:
  $L(FPHP(G) \vdash \perp) = \exp\left(\Omega\left(\frac{n^2}{m \log m}\right)\right)$



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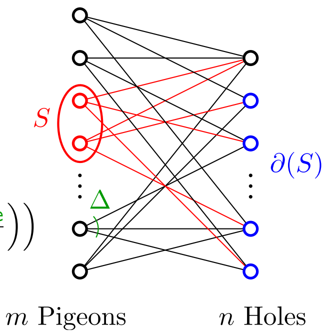

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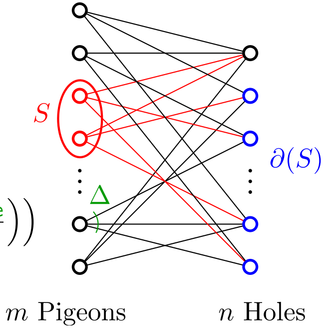
  $L(\mathit{Onto}\text{-}FPHP(G) \vdash \perp) = \exp\left(\Omega\left(\frac{\text{min degree}}{\log^2 m}\right)\right)$
- What about $m \gg n^2$ and $\Delta \approx \log m$?



Our Weak Graph PHP Lower Bounds

- For $m \leq n^{\mathrm{o}(\log n)}$, $\Delta = \log m$ and G sampled from $\mathcal{G}(m, n, \Delta)$:

$$L(FPHP(G) \vdash \perp) \geq \exp\left(n^{1-o(1)}\right)$$

Our Weak Graph PHP Lower Bounds

- For $m \leq n^{\mathrm{o}(\log n)}$, $\Delta = \log m$ and G sampled from $\mathcal{G}(m, n, \Delta)$:

$$L\big(FPHP(G) \vdash \bot\big) \geq \exp\left(n^{1-o(1)}\right)$$

- For $m = n^k$, $\Delta = 32\left(\frac{k}{\varepsilon}\right)^2$ and G sampled from $\mathcal{G}(m, n, \Delta)$:

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- For $m < \exp\left(n^{1/16}\right)$ and $\Delta = \mathcal{O}(\mathrm{polylog}(m))$, $\exists$ graphs G such that:

$$L(FPHP(G) \vdash \bot) \geq \exp\left(n^{1/5}\right)$$

(using expander construction in [GUV09])

A General Theorem

Theorem

Let G be an $(r, \Delta, (1 - \varepsilon \log n / \log m)\Delta)$-boundary expander. Then

$$L(FPHP(G) \vdash \bot) = \exp\left(\Omega\left(\frac{r}{n^{\varepsilon} \log^2 m}\right)\right)$$

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Technical note:

- Need expansion $\lim_{n\to\infty} c = \Delta$
- Would be great to show that $c = (1 - \varepsilon)\Delta$ is enough
- Probably room for improvement also in other parameters

Very High-Level Proof Outline

- Define pseudo-width measure on clauses $\approx$ interesting pigeons

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- Short refutations can be transformed into low-width refutations

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- But any refutation of $FPHP(G)$ requires large pseudo-width
- Hence, no short refutations can exist $\blacksquare$

Pseudo-Width

Each clause has 3 different kinds of pigeons:

Pseudo-Width

Each clause has 3 different kinds of pigeons:

obese

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Pseudo-Width

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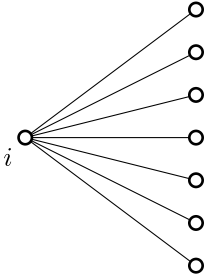
slim

Measuring the Strength of Clauses

- How strong is a clause C?



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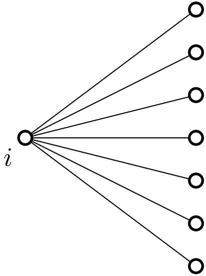

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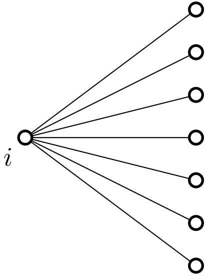

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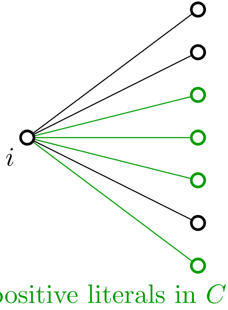

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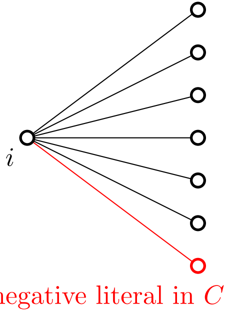

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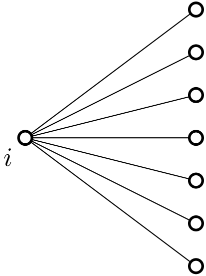

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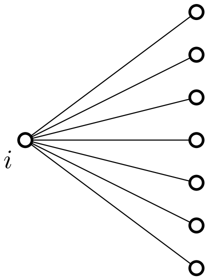

Key take-away

For each pigeon, consider #matchings that C rules out

Obese Pigeons

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- $P_{\mathrm{obese}}(C) = \{i \in [m] \mid d_i(C) \geq d_i\}$

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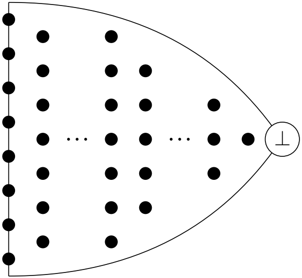
- $P_{\mathrm{stout}}(C) = \{i \in [m] \mid d_i(C) \geq d_i - \delta\}$
- Pseudo-width of clause C is

$$W^*(C) = \left|P_{\mathrm{stout}}(C)\right|$$

i.e., #stout pigeons in C

Refined Proof Outline

1. Given refutation, classify clauses as having high or low pseudo-width



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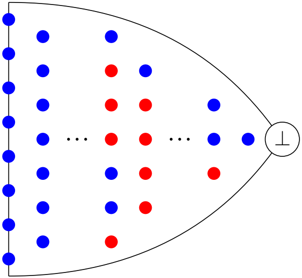

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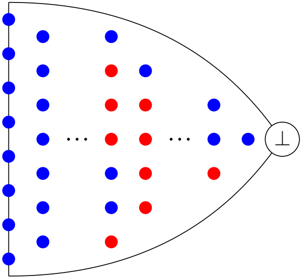

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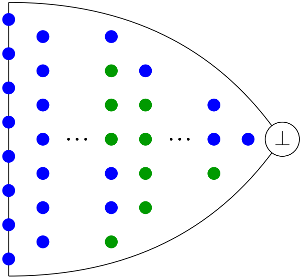

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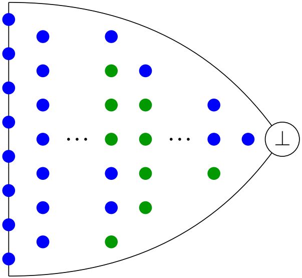

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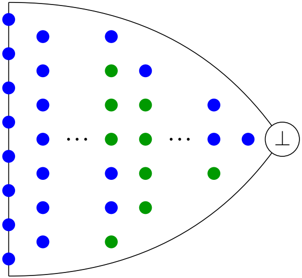

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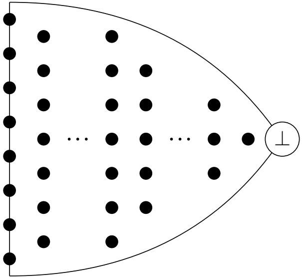
4. Show that since $\mathcal{A}$ not too large, $FPHP(G) \cup \mathcal{A}$ must still require large pseudo-width ↯ ■

Filter Lemma

Lemma (Razborov [Raz03] (with a small twist))

If $\delta \leq \varepsilon \frac{\Delta \log n}{\log m}$ *and length* $L < 2^{w_0}$*, then* $\exists\ \vec{d} = (d_1, \ldots, d_m)$ *such that* $\forall$ *clauses* C *in refutation one of two cases applies:*

1. $|P_{\text{obese}}(C)| \geq w_0$
2. $|P_{\text{stout}}(C)| = W^*(C) \leq \mathcal{O}(w_0 \cdot n^{\varepsilon})$



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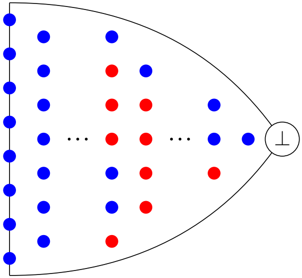

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- *filter vector $\vec{d}$*
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Proof:

1. Replace type-1 clauses with many obese pigeons by (stronger) fake axioms
2. Now all clauses have low width (type-2 clauses were already OK) — done!

Pseudo-Width Lower Bound: Statement and Intuition

Lemma

Suppose G is $(r, \Delta, (1 - \varepsilon \log n / \log m)\Delta)$-boundary expander. Then refuting $FPHP(G) \cup \mathcal{A}$ requires pseudo-width $\Omega(r \cdot \log n / \log m)$

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- $\Rightarrow$ Too few fake axioms to add up to 100% $\lightning$

More About the Actual Pseudo-Width Lower Bound

A couple of issues:

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3. Do proof on previous slide, but with linear algebra ☺

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Main Technical Lemma

For derivations in low pseudo-width it holds that

$$Z(C \vee D) \subseteq \mathrm{span}\big(Z(C \vee x), Z(D \vee \overline{x})\big)$$

Take-Home Message

- Resolution very well-studied; large toolbox developed
- But many challenging problems remain beyond current techniques
- Razborov's pseudo-width method seems like a powerful tool that might also work for, e.g.,
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Thanks! Questions?

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