A Generalized Method for Proving Polynomial Calculus Degree Lower Bounds

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Joint work with Mladen Mikša

The Satisfiability Problem (SAT)

$$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$$

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- $\land$ means all constraints should hold simultaneously

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$$(x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z})$$

- Variables should be set to true or false
- Constraint $(x \vee \overline{y} \vee z)$: means x or z should be true or y false
- $\wedge$ means all constraints should hold simultaneously

Is there a truth value assignment satisfying all these conditions?
Or is it always the case that some constraint must fail to hold?

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Satisfiable formulas have short, efficiently verifiable certificates (satisfying assignments)

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Proof system

Formal specification of method for reasoning about formulas
Given formula $\mathcal{F}$, can produce certificate π of unsatisfiability
Proof π should be polynomial-time verifiable (in size of π, not $\mathcal{F}$)

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Proof complexity

Study of upper and lower bounds for concrete proof systems

Motivations for Proof Complexity

Program for showing P $\neq$ NP

Original motivation in [Cook & Reckhow '79]

Superpolynomial lower bounds for all proof systems $\Rightarrow$ NP $\neq$ co-NP

Still very distant goal...

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Quantify power of mathematical reasoning

Study efficient proofs of different mathematical principles

Determine how strong proof systems are needed

Measures "mathematical depth" of corresponding principle

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Connections to SAT solving and combinatorial optimization

Can formalize and study proof systems behind state-of-the-art SAT solvers

Sheds light on potential and limitations of such solvers

Also extends to combinatorial optimization (e.g., LP and SDP hierarchies)

Outline of This Presentation

1. Overview of some proof complexity basics
2. Discuss two proof systems
   - Resolution ($\Leftrightarrow$ state-of-the-art conflict-driven clause learning solvers)
   - Polynomial calculus ($\Leftrightarrow$ algebraic Gröbner basis computations)
3. Present framework for proving polynomial calculus lower bounds
   - Based on degree lower bounds via expansion
   - Expressed in terms of combinatorial game played on formula
   - Unifies previous lower bounds and yields some new ones

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (where k is some constant)
- $N =$ size of formula (# literals, which is $\approx$ # clauses for k-CNF)

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

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Can represent refutation as

- annotated list or
- directed acyclic graph

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6.	$x \lor \overline{y}$	Res$(2, 4)$
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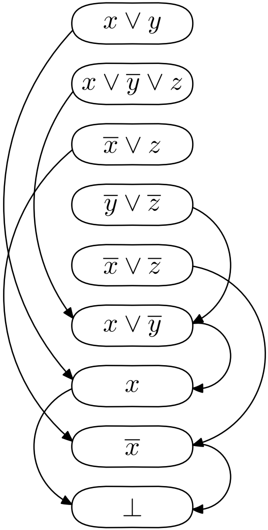

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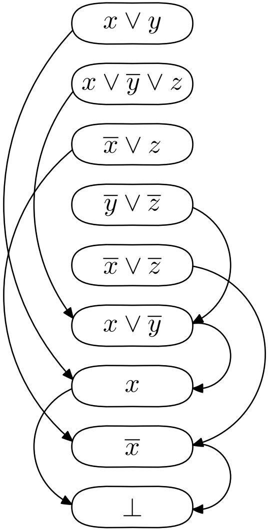
Tree-like resolution if DAG is tree



Resolution Size/Length

Size/length $= \#$ clauses in refutation

Most fundamental measure in proof complexity

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known

Examples of Hard Formulas w.r.t Resolution Size (1/2)

Pigeonhole principle (PHP) [Haken '85]
"$n + 1$ pigeons don't fit into n holes"

Variables $p_{i,j} =$ "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
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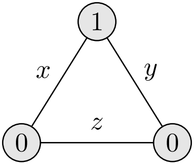
Even onto functional PHP formula is hard for resolution
"Resolution cannot count"

Examples of Hard Formulas w.r.t Resolution Size (2/2)

Tseitin formulas [Urquhart '87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



$$
\begin{aligned}
&(x \lor y) && \land\ (\overline{x} \lor z) \\
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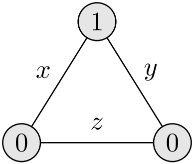

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Requires size $\exp(\Omega(N))$ on well-connected so-called expanders
"Resolution cannot count $\mathrm{mod}\ 2$"

Resolution Width

Width = size of largest clause in refutation (always $\leq N$)

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Width upper bound $\Rightarrow$ size upper bound

Proof: at most $(2 \cdot \#\text{variables})^{\text{width}}$ distinct clauses
(This simple counting argument is essentially tight [Atserias et al.'14])

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Width lower bound $\Rightarrow$ size lower bound

Much less obvious...

Width Lower Bounds Imply Size Lower Bounds

Theorem ([Ben-Sasson & Wigderson '99])

$$\mathit{size} \geq \exp\left(\Omega\left(\frac{(\mathit{width})^2}{(\mathit{formula\ size}\ N)}\right)\right)$$

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For tree-like resolution have size $\geq 2^{\text{width}}$ [Ben-Sasson & Wigderson '99]

General resolution: width up to $\mathcal{O}\left(\sqrt{N \log N}\right)$ implies no size lower bounds — possible to tighten analysis? **No!**

Optimality of the Size-Width Lower Bound

Ordering principles [Stålmarck '96, Bonet & Galesi '99]
"Every (partially) ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j} =$ "$e_i < e_j$"

$\overline{x}_{i,j} \vee \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \vee \overline{x}_{j,k} \vee x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
$\bigvee_{1 \leq i \leq n,\, i \neq j} x_{i,j}$	e_j is not a minimal element

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Refutable in resolution in size $\mathcal{O}(N)$
Requires resolution width $\Omega\big(\sqrt[3]{N}\big)$ (converted to 3-CNF)

Polynomial Calculus

Introduced in [Clegg et al. '96]; slightly modified in [Alekhnovich et al. '00]

Clauses interpreted as polynomial equations over finite field
Any field in theory; $\mathrm{GF}(2)$ in practice
Example: $x \vee y \vee \overline{z}$ gets translated to $xy\overline{z} = 0$
(Think of $0 \equiv true$ and $1 \equiv false$)

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Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$

Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \quad q = 0}{\alpha p + \beta q = 0}$

Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

Polynomial Calculus Size and Degree

Clauses turn into monomials
Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

Polynomial Calculus Size and Degree

Clauses turn into monomials
Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

Size — analogue of resolution length/size
total # monomials in refutation counted with repetitions

Degree — analogue of resolution width
largest degree of monomial in refutation

Polynomial Calculus Can Mimic Resolution Steps

Example: Resolution step:

$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

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simulated by polynomial calculus derivation:

$$\dfrac{x\overline{y}z = 0 \qquad \dfrac{\dfrac{\overline{y}\overline{z} = 0}{x\overline{y}\overline{z} = 0} \qquad \dfrac{\dfrac{z + \overline{z} - 1 = 0}{\overline{y}z + \overline{y}\overline{z} - \overline{y} = 0}}{x\overline{y}z + x\overline{y}\overline{z} - x\overline{y} = 0}}{-x\overline{y}z + x\overline{y} = 0}}{x\overline{y} = 0}$$

Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus simulates resolution efficiently

- Can mimic refutation step by step as shown on previous slide
- Essentially no increase in length/size or width/degree
- Hence worst-case upper bounds for resolution carry over

Polynomial calculus is strictly stronger w.r.t. both size and degree

- Consider, e.g., Tseitin formulas on expanders
- Over $\mathrm{GF}(2)$ can just do Gaussian elimination
- Also other examples not depending on field characteristic

Size vs. Degree

- Degree upper bound ⇒ size upper bound [Clegg et al.'96]
  Qualitatively similar to resolution bound
  A bit more involved argument
  Again essentially tight by [Atserias et al.'14]

- Degree lower bound ⇒ size lower bound [Impagliazzo et al.'99]
  Precursor of [Ben-Sasson & Wigderson '99] — can do same proof to get same bound

- Size-degree lower bound essentially optimal [Galesi & Lauria '10]
  Example: same ordering principle formulas

- Most size lower bounds for polynomial calculus derived via degree lower bounds (but machinery much less developed)

Lower Bounds via Expansion

Standard approach: Lower bounds from expansion

Simplest example: Clause-variable incidence graph (CVIG)

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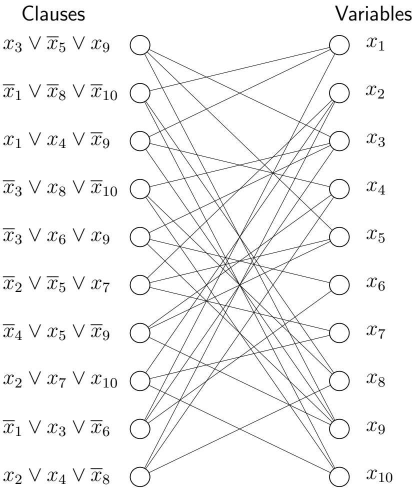

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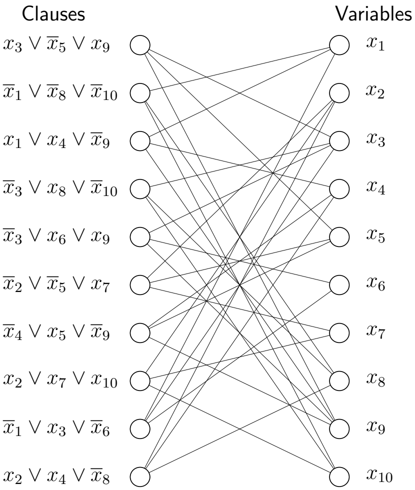
Subsets of left vertices have many unique neighbours on right



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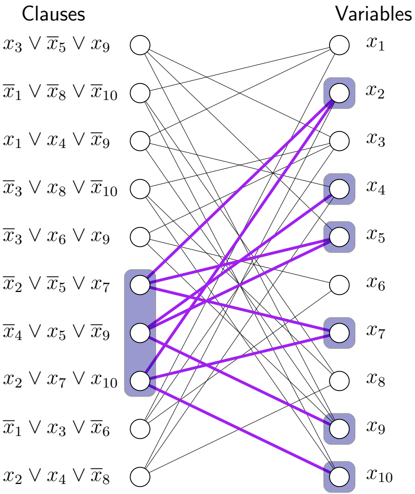

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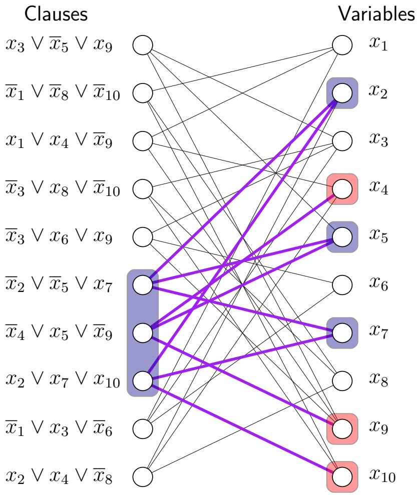

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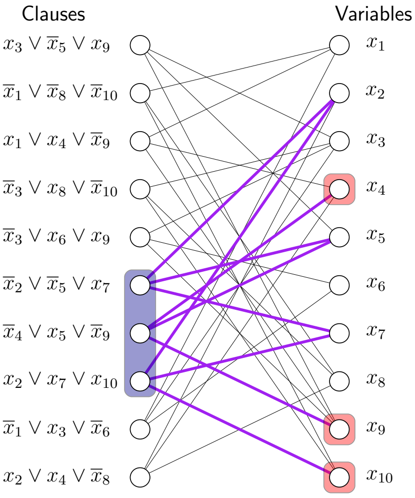

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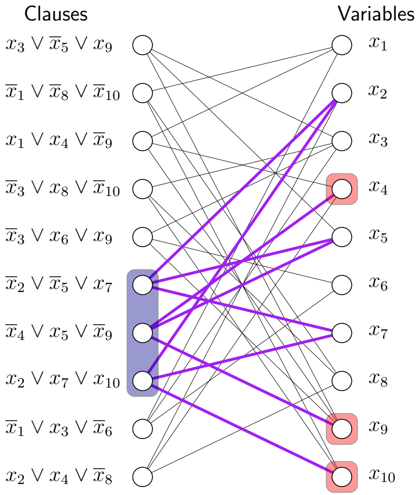

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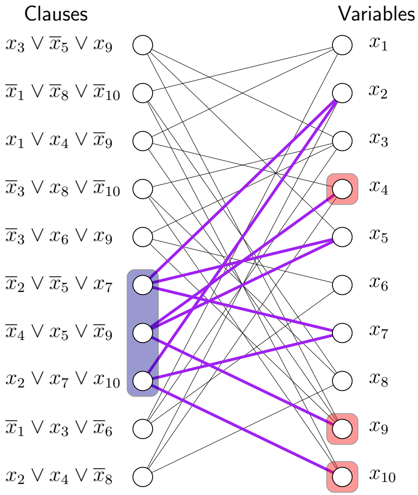
Need graph capturing underlying principle!



Our Results

Main Theorem (Informal)

Graph structure on formula such that expansion implies hardness in polynomial calculus

Extends an approach from [Alekhnovich, Razborov '01]

Unifies many previous lower bounds for polynomial calculus

Corollary: New lower bound resolving open question in [Razborov '02]

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Warm-up: Use resolution to present main ideas and challenges

Revisiting Tseitin Formulas

Given set of equations over $\mathbb{F}_2$

$$\begin{aligned} x + w &= 0 \\ x + y &= 0 \\ y + w + z &= 1 \\ z &= 0 \end{aligned}$$

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Encode as clauses

Clauses

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$\overline{x} \lor w$

$x \lor \overline{y}$

$\overline{x} \lor y$

$y \lor w \lor z$

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$\overline{z}$

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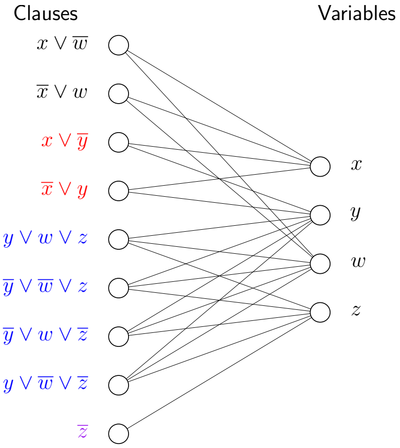

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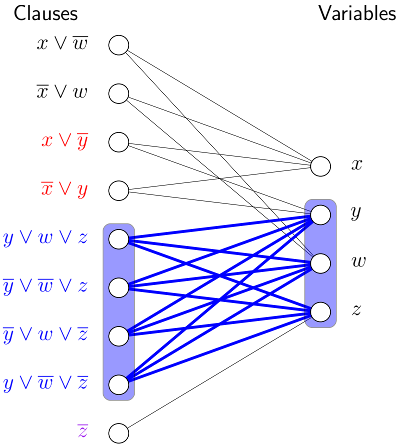

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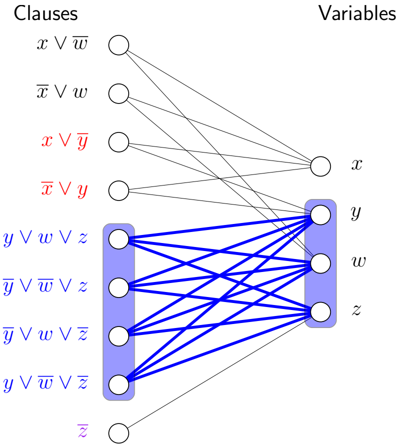
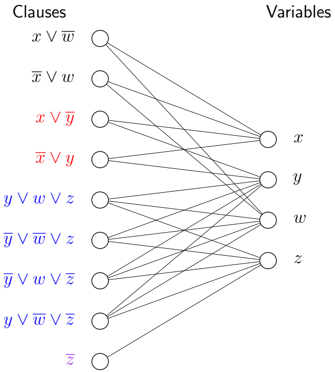
Graph should encode equations, not clauses!

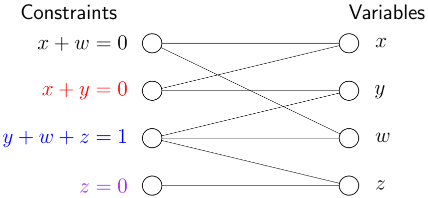


Constraint-Variable Incidence Graph

Use one vertex per equation on the left

Put edge if variable appears in equation





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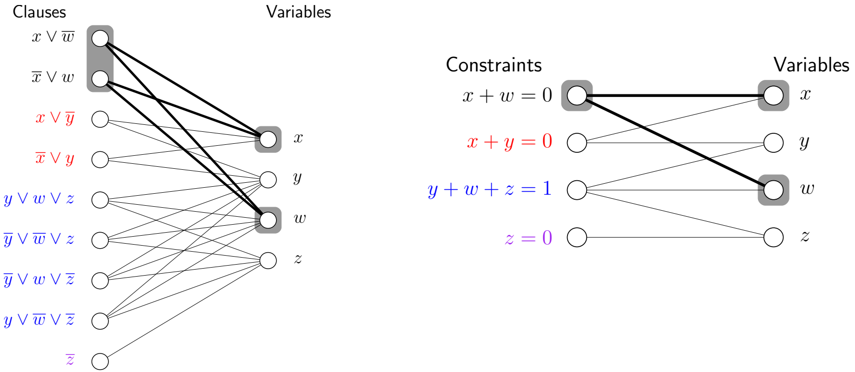

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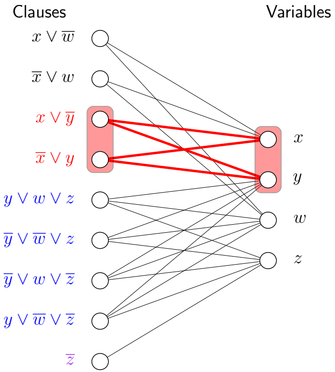

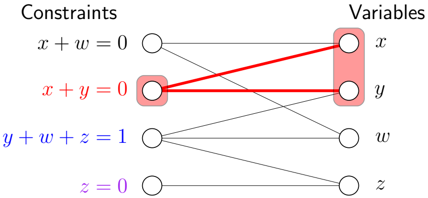

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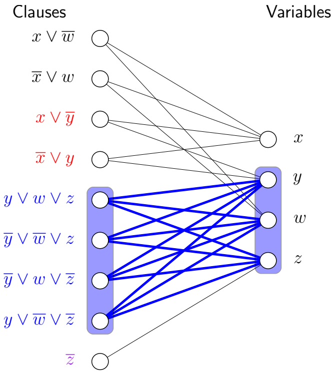

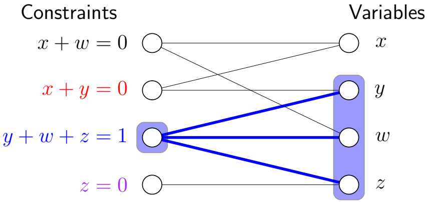

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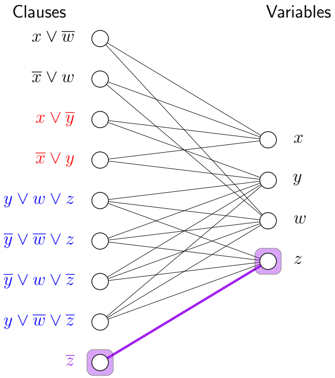

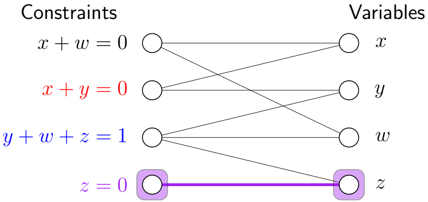

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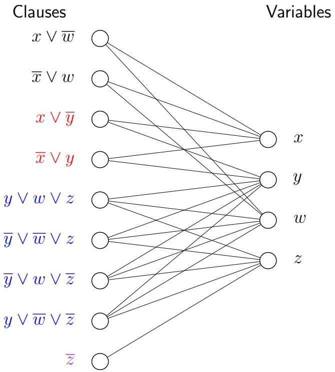

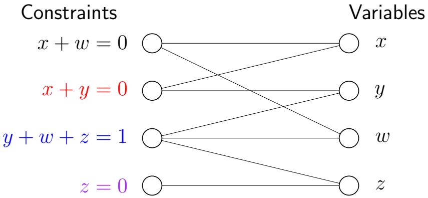

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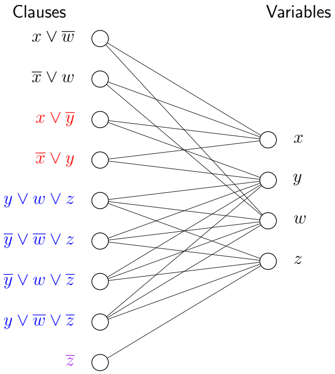
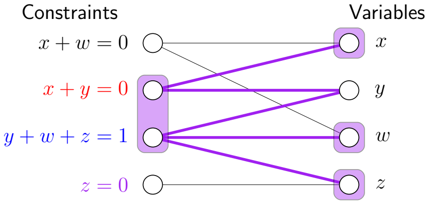
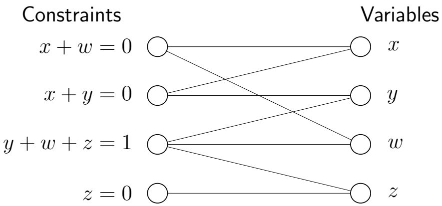




Now the constraint-variable incidence graph expands!

Proof Sketch of Tseitin Lower Bound



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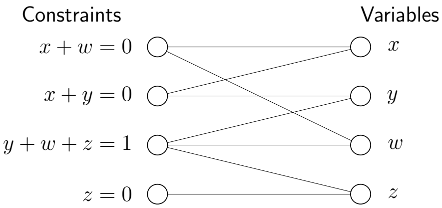

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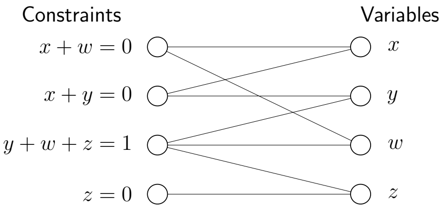

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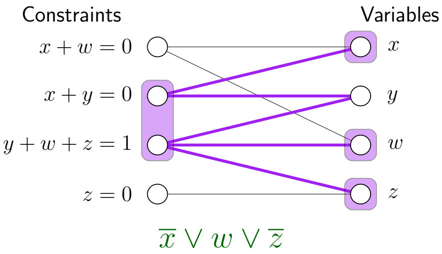

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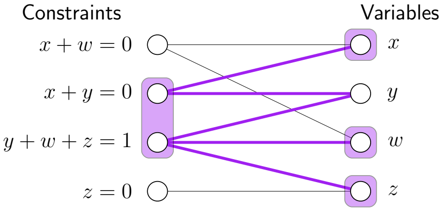

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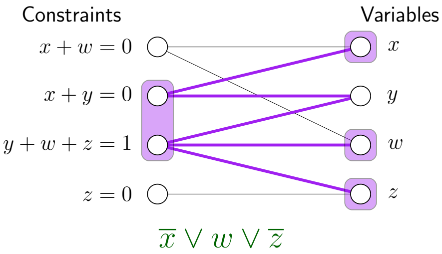

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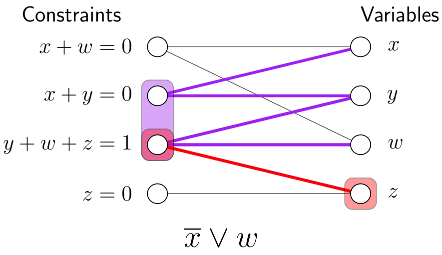

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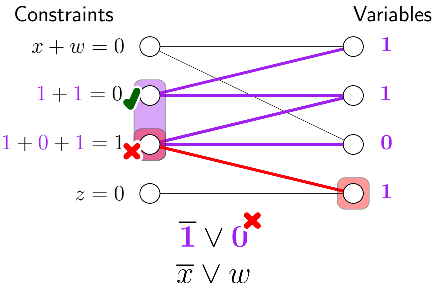

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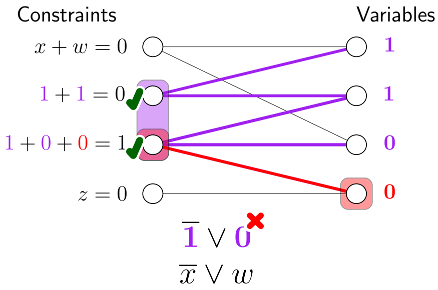

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Resolution Lower Bounds

Resolution edge game on (P, x)

1. Adversary provides assignment ρ to all variables
2. Can flip x to some b so that P is satisfied

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Theorem (Ben-Sasson & Wigderson '99)

If from formula $\mathcal{F} = \bigwedge_{P \in \mathcal{F}} P$ can form bipartite graph $\mathcal{G}(\mathcal{F})$ such that

- *$\mathcal{G}(\mathcal{F})$ is expanding and*
- *for all edges (P, x) we can satisfy P by flipping x*

then refuting $\mathcal{F}$ requires large width

Polynomial Calculus Edge Game

Tseitin: linear equations $\Rightarrow$ easy over $\mathbb{F}_2$ (Gaussian elimination)

Need stronger guarantee from constraint-variable incidence graph!

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For polynomial calculus we have to play a harder game

Polynomial calculus edge game on (P, x)

1. Commit to assignment $x = b$ ahead of time
2. Adversary provides assignment ρ to all variables
3. Flipping $x = b$ satisfies P

Easy to see we can't win this game for Tseitin formulas

Main Theorem (Preliminary Version)

If from formula $\mathcal{F} = \bigwedge_{P \in \mathcal{F}} P$ can form bipartite graph $\mathcal{G}(\mathcal{F})$ such that:

- $\mathcal{G}(\mathcal{F})$ is expanding, and
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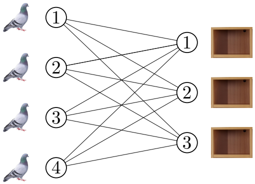
Not enough to prove functional pigeonhole principle hard!

Pigeonhole Principle (PHP)

"$n + 1$ pigeons don't fit into n holes"



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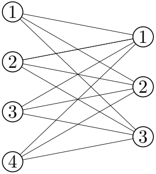
Pigeons

$x_{1,1} \lor x_{1,2} \lor x_{1,3}$

$x_{2,1} \lor x_{2,2} \lor x_{2,3}$

$x_{3,1} \lor x_{3,2} \lor x_{3,3}$

$x_{4,1} \lor x_{4,2} \lor x_{4,3}$



Holes

$\overline{x}_{1,1} \lor \overline{x}_{2,1}$ $\quad$ $\overline{x}_{2,1} \lor \overline{x}_{3,1}$
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Very wide clauses — hit with restriction to decrease width
Restricts choices of holes for each pigeon — graph PHP formula

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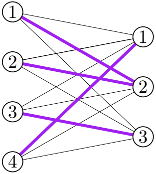

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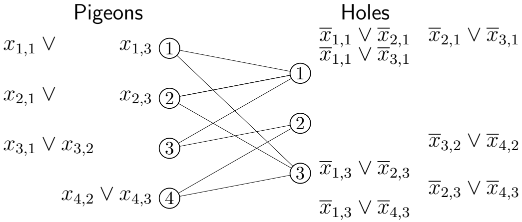

Pigeonhole Principle (PHP)

"$n+1$ pigeons don't fit into n holes"

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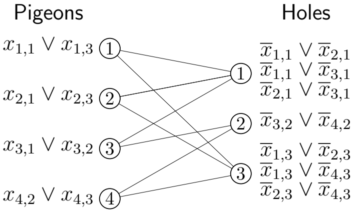

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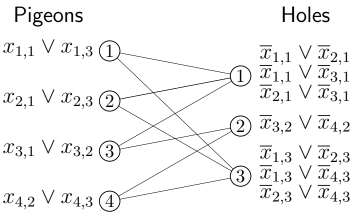

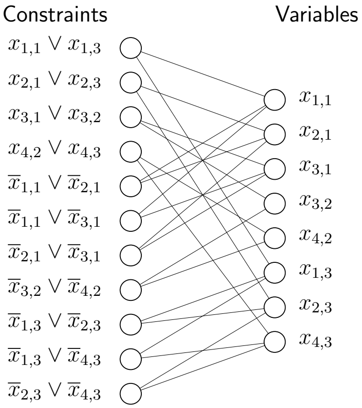

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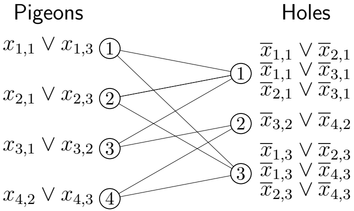

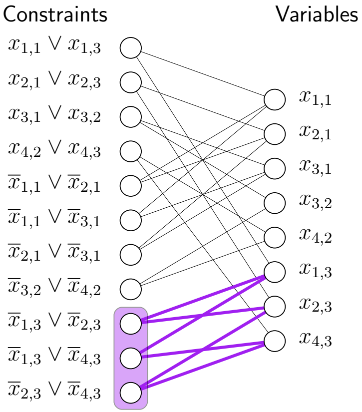

But again CVIG not expanding!

Proving PHP Lower Bound

Isolate hole axioms from graph and group hole variables together

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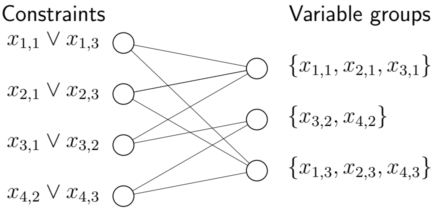

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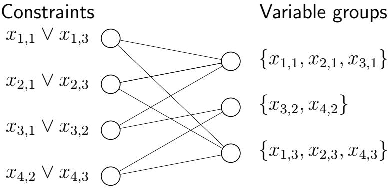

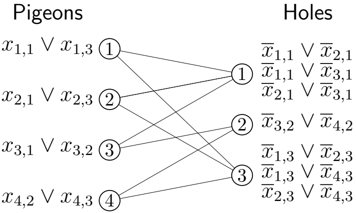

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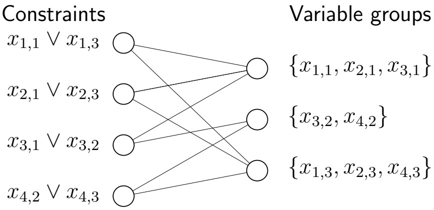

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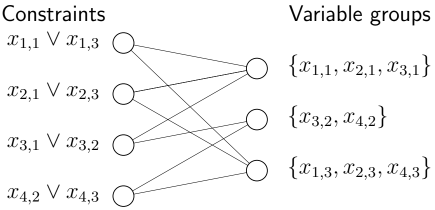

- Change game to play on assignments to groups of variables
- Assignments must satisfy any hole axioms touched

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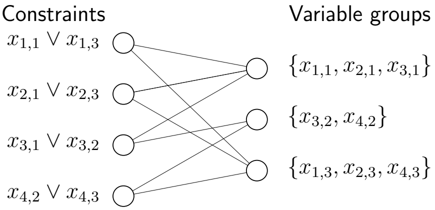

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Polynomial calculus edge game on (P, V) with side constraint E

1. Commit to $\rho_V : V \to \{0, 1\}$ satisfying any clauses touched in E
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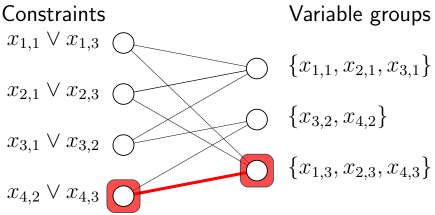

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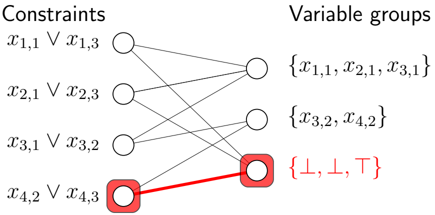

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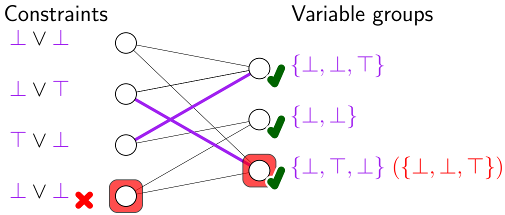

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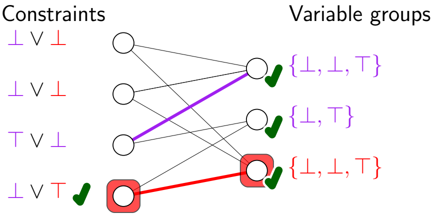

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Generalized Method for Degree Lower Bounds

Main Theorem

If from formula $\mathcal{F} = \mathcal{F}' \wedge E$ we can form bipartite graph $\mathcal{G}(\mathcal{F}')$ such that:

- $\mathcal{G}(\mathcal{F}')$ is expanding, and
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then refuting $\mathcal{F}$ requires large degree

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Provides common framework for previous lower bounds:

- CNF formulas with expanding CVIGs [Alekhnovich & Razborov '01]
- Pigeonhole principle [Alekhnovich & Razborov '01]
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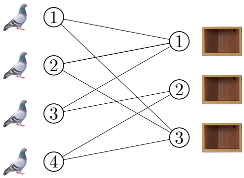
Allows us to establish that functional PHP is hard

PHP Variants



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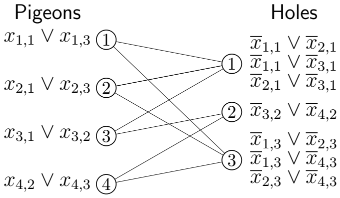

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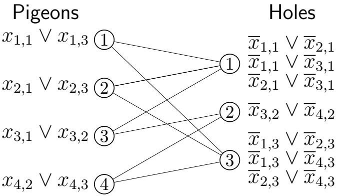

- Can have “fat pigeons” assigned to multiple holes

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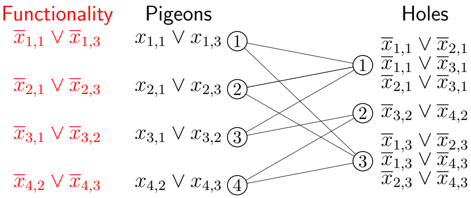

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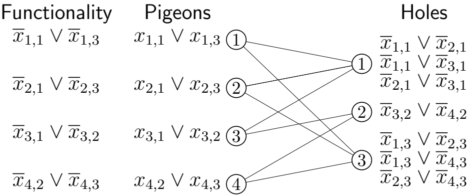

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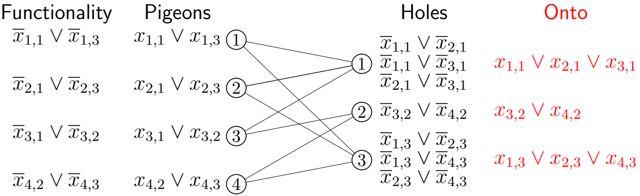

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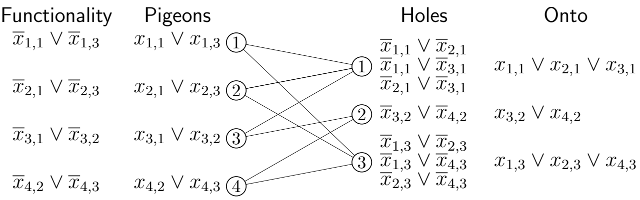

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PHP Variants

Functionality

$\overline{x}_{1,1} \lor \overline{x}_{1,3}$

$\overline{x}_{2,1} \lor \overline{x}_{2,3}$

$\overline{x}_{3,1} \lor \overline{x}_{3,2}$

$\overline{x}_{4,2} \lor \overline{x}_{4,3}$

Pigeons

$x_{1,1} \lor x_{1,3}$

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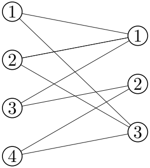

Holes

$\overline{x}_{1,1} \lor \overline{x}_{2,1}$
$\overline{x}_{1,1} \lor \overline{x}_{3,1}$
$\overline{x}_{2,1} \lor \overline{x}_{3,1}$

$\overline{x}_{3,2} \lor \overline{x}_{4,2}$

$\overline{x}_{1,3} \lor \overline{x}_{2,3}$
$\overline{x}_{1,3} \lor \overline{x}_{4,3}$
$\overline{x}_{2,3} \lor \overline{x}_{4,3}$

Onto

$x_{1,1} \lor x_{2,1} \lor x_{3,1}$

$x_{3,2} \lor x_{4,2}$

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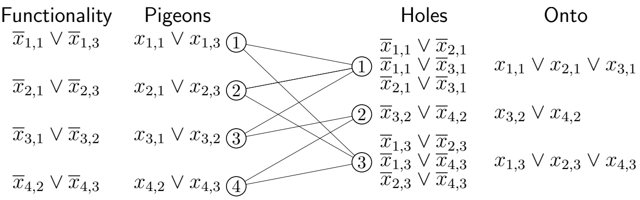
Functional PHP = PHP + Functionality

PHP Variants



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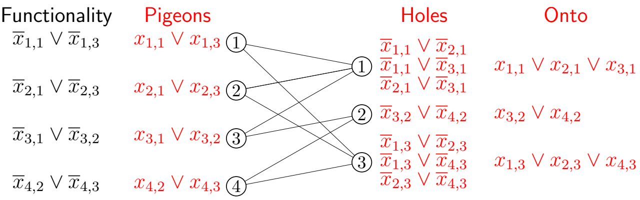

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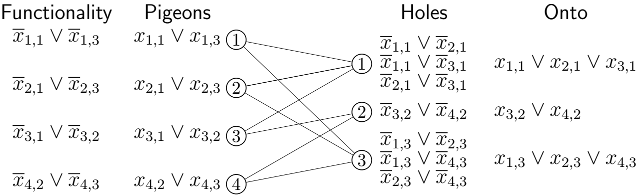

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PHP Variants

Functionality	Pigeons
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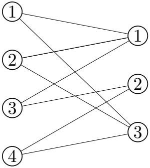

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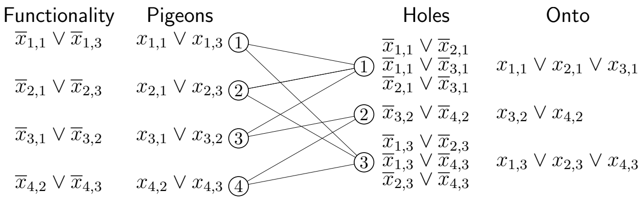
Onto-FPHP = PHP + Functionality + Onto



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Hardness of PHP Variants

Variant	Resolution	Polynomial calculus
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FPHP		
Onto-PHP		
Onto-FPHP		

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PHP	hard [Haken '85]	
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This work

- Observe that [AR '01] proves hardness of Onto-PHP
- Prove that FPHP is hard in polynomial calculus

Open Problems

- Prove polynomial calculus lower bounds for other formulas

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  - Cannot deal with lower bounds à la [Buss et al. '99]
- Go beyond polynomial calculus (to sums-of-squares, for instance)

Take-away Message

Generalized method for polynomial calculus degree lower bounds

- Unified framework for most previous lower bounds
- Exponential size lower bound for Functional PHP

Future directions

- Extend techniques further to other tricky formulas
- Develop non-degree-based size lower bound techniques

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Thank you for your attention!