A Generalized Method for Proving Polynomial Calculus Degree Lower Bounds

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Joint work with Mladen Mikša

The Satisfiability Problem (SAT)

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1. Can this problem be solved efficiently?
2. Is there an efficiently verifiable certificate for correct answer?

SAT and Proof Complexity

SAT, NP, and co-NP

- SAT NP-complete [Cook '71, Levin '73], hence unlikely to be solvable efficiently worst-case
- Satisfiable formulas have small certificates (assignment)
- Unsatisfiable formulas don't, unless NP $=$ co-NP
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Proof complexity

- Prove lower bounds on certificate size for increasingly stronger formal methods of reasoning ($\approx$"separation NP $\neq$ co-NP in weak computational models")
- Analyze algorithms used in practice for SAT solving
- Quantify hardness/depth of different mathematical theorems

Proof Complexity and Expansion

- **General goal:** Prove that concrete proof systems cannot efficiently certify unsatisfiability of concrete CNF formulas
- **General theme:**

CNF formula $\mathcal{F}$ "expanding"

$\Downarrow$

Large proofs needed to refute $\mathcal{F}$

- Well-developed machinery for resolution
- Very much less so for polynomial calculus
- What "expanding" means is usually a formula-specific hack

A General Expansion Criterion for Hardness

Given CNF formula $\mathcal{F}$ over variables $\mathcal{V}$, build bipartite graph

- Left vertex set partition of clauses into $\mathcal{F} = \bigcup_{i=1}^{m} F_i$
- Right vertex set division of variables $\mathcal{V} = \bigcup_{j=1}^{n} V_j$
- Edge (F_i, V_j) if $Vars(F_i) \cap V_j \neq \emptyset$

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- Adversary assigns all variables $\mathcal{V} \setminus V_j$
- We assign V_j
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Consequences

- Extends [Ben-Sasson & Wigderson '99] and [Alekhnovich & Razborov '01]
- Unifies many previous lower bounds
- Corollary: Lower bound resolving problem in [Razborov '02]

Outline

1. Proof Complexity Overview
   - Preliminaries
   - Resolution
   - Polynomial Calculus
2. Lower Bounds from Expansion
   - Resolution Width
   - Polynomial Calculus Degree
   - Pigeonhole Principle
3. Open Problems

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $\mathcal{F} = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  $k = \mathcal{O}(1)$ constant in this talk
- M = size of formula = # literals ($\approx$ # clauses for k-CNF)
- N = # variables $\leq M$

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

1. $x \lor y$
2. $x \lor \overline{y} \lor z$
3. $\overline{x} \lor z$
4. $\overline{y} \lor \overline{z}$
5. $\overline{x} \lor \overline{z}$

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2.	$x \vee \overline{y} \vee z$	Axiom
3.	$\overline{x} \vee z$	Axiom
4.	$\overline{y} \vee \overline{z}$	Axiom
5.	$\overline{x} \vee \overline{z}$	Axiom
6.	$x \vee \overline{y}$	Res$(2, 4)$
7.	x	Res$(1, 6)$
8.	$\overline{x}$	Res$(3, 5)$
9.	$\perp$	Res$(7, 8)$

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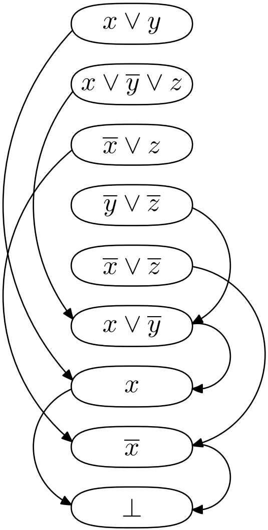

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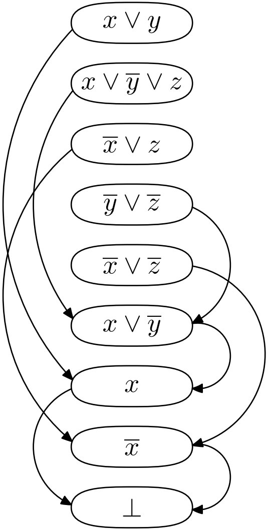
Tree-like resolution if DAG is tree



Resolution Size/Length

Size/length $= \#$ clauses in refutation

Most fundamental measure in proof complexity

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(M))$ lower bounds known

(Recall $N = \#$ variables $\leq$ formula size $= M$)

Examples of Hard Formulas w.r.t Resolution Size (1/2)

Pigeonhole principle (PHP) [Haken '85]
"$n+1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

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But only lower bound $\exp\left(\Omega\left(\sqrt[3]{M}\right)\right)$ in terms of formula size

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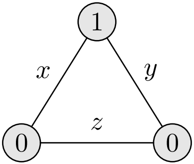
Tseitin formulas [Urquhart '87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



$$
\begin{aligned}
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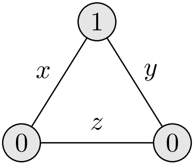

$$
\begin{array}{rr}
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Requires size $\exp(\Omega(M))$ on bounded-degree edge expanders
"Resolution cannot count $\bmod 2$**"**

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Much less obvious. . .

Width Lower Bounds Imply Size Lower Bounds

Theorem ([Ben-Sasson & Wigderson '99])

For k-CNF formula over N variables

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For tree-like resolution have proof size $\geq 2^{\text{width}}$ [BW99]

General resolution: width up to $\mathcal{O}\left(\sqrt{N \log N}\right)$ implies no size lower bounds — possible to tighten analysis? **No!**

Optimality of the Size-Width Lower Bound

Ordering principles [Stålmarck '96, Bonet & Galesi '99]
"Every (partially) ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j} =$ "$e_i < e_j$"

$\overline{x}_{i,j} \lor \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
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Refutable in resolution in size $\mathcal{O}\left(N^{3/2}\right) = \mathcal{O}(M)$
Requires resolution width $\Omega\left(\sqrt{N}\right)$ (converted to k-CNF)

Conversion to k-CNF "Graph Versions" of Formulas

- Need bounded-width CNFs to use lower bound in [BW99]
- But PHP and ordering principle formulas have wide clauses
- **Solution:** Restrict formulas to bounded-degree graphs

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- Now width lower bounds $\Rightarrow$ size lower bounds
- And size lower bounds hold for original, unrestricted formulas

Polynomial Calculus (PC)

From [Clegg et al. '96] with adjustment in [Alekhnovich et al. '02]

Clauses interpreted as polynomial equations over field

Example: $x \vee y \vee \overline{z}$ gets translated to $xy\overline{z} = 0$
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Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$ Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \quad q = 0}{\alpha p + \beta q = 0}$ Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

Polynomial Calculus Size and Degree

Clauses turn into monomials
Write out all polynomials as sums of monomials
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Size — analogue of resolution length/size
total # monomials in refutation counted with repetitions

Degree — analogue of resolution width
largest degree of monomial in refutation

Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus simulates resolution efficiently

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Polynomial calculus strictly stronger w.r.t. size and degree

- Tseitin formulas (over $\mathrm{GF}(2)$ can do Gaussian elimination)
- Onto functional pigeonhole principle (over any field) [Riis '93]
- Also other examples

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- **Open problem:** Are functional PHP and onto PHP formulas hard for polynomial calculus?

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Lower bounds from expansion

Simplest example is the clause-variable incidence graph (CVIG)

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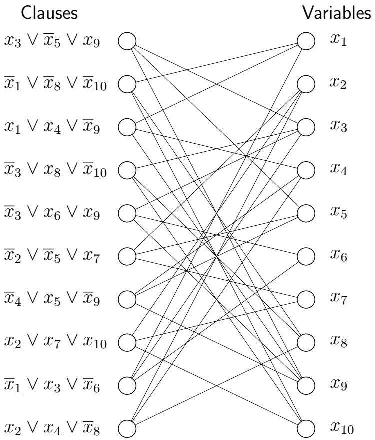

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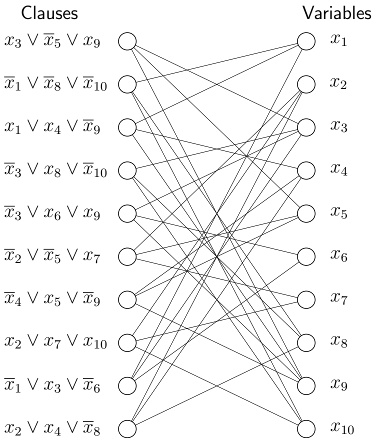

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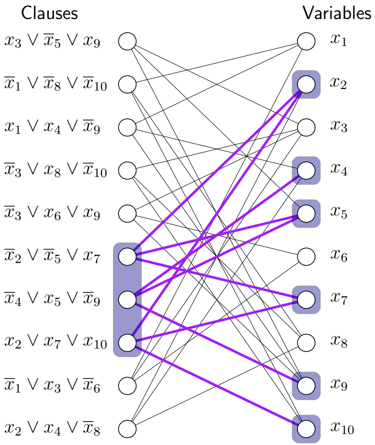

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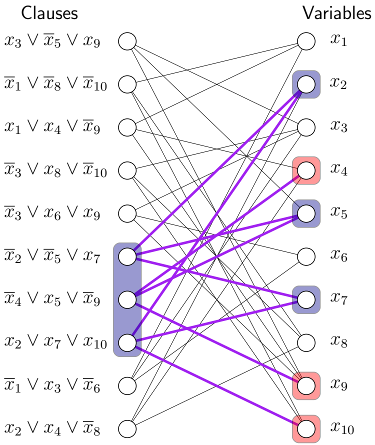

Lower Bounds via Graph Expansion

Standard approach:
Lower bounds from expansion

Simplest example is the clause-variable incidence graph (CVIG)

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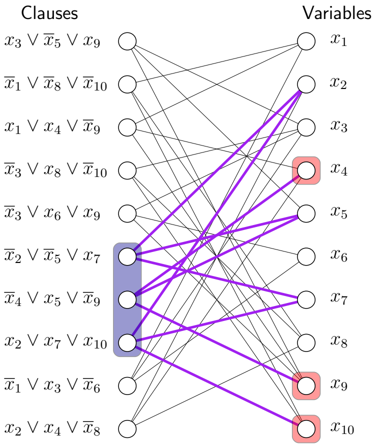

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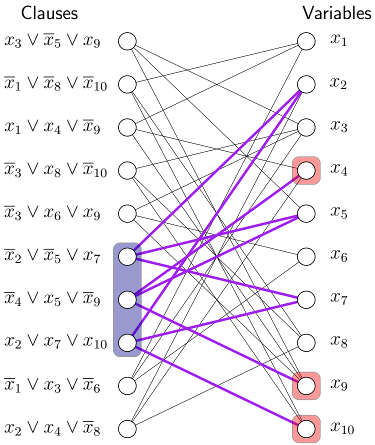

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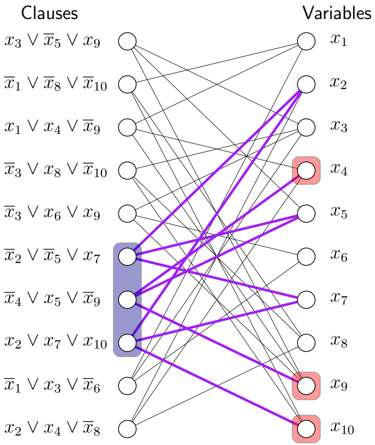
Need graph capturing combinatorial structure!



Generalized Incidence Graphs for CNF Formulas

Given CNF formula $\mathcal{F}$ over variables $\mathcal{V}$

- Partition clauses into $\mathcal{F} = E \cup \bigcup_{i=1}^{m} F_i$ (for E satisifiable)
- Divide variables into $\mathcal{V} = \bigcup_{j=1}^{n} V_j$ — **not** always partition
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Build bipartite $(\mathcal{U}, \mathcal{V})_E$-graph $\mathcal{G}$

- Left vertices $\mathcal{U} = \{F_1, \ldots, F_m\}$
- Right vertices $\mathcal{V} = \{V_1, \ldots, V_n\}$
- Edge (F_i, V_j) if $\mathit{Vars}(F_i) \cap V_j \neq \emptyset$
- Two types of edges depending on how F_i and V_j behave (modulo assignments α satisfying "filtering set" E)

The Importance of Basic Courtesy

$F \in \mathcal{U}$ and $V \in \mathcal{V}$ are E-semirespectful neighbours if

- given any total assignment α such that $\alpha(E) = 1$
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Example

$F_1 = \{x \vee y,\ x \vee \overline{z},\ \overline{x} \vee z\}$, $V = \{x, y\}$, $E = \{\overline{y} \vee z\}$

Not E-semirespectful — consider $\alpha = \{y \mapsto 0, z \mapsto 0\}$

Not allowed to flip $z \notin V$; flipping y falsifies E; but $F_1\restriction_\alpha = \{x, \overline{x}\}$

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Change to $F_2 = \{x \vee \overline{y}, x \vee \overline{z}, \overline{x} \vee y \vee z\}$, $V = \{x, y\}$, $E = \{\overline{y} \vee z\}$

Now F_2 and V E-semirespectful — given any α s.t. $\alpha(\overline{y} \vee z) = 1$

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(To simplify, think of all edges (F_i, V_j) as being E-semirespectful)

Semirespectful Expanders and Width Lower Bounds

Recall boundary $\partial(\mathcal{U}') = \{V \in \mathcal{N}(\mathcal{U}') \mid \mathcal{N}(V) \cap \mathcal{U}' = \{F\} \text{ unique}\}$

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Theorem (essentially [BW99])

If $\mathcal{F}$ has (s, δ, E)-semirespectful expander $(\mathcal{U}, \mathcal{V})_E$ with overlap ℓ, then

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Progress Measure Approach (1/4)

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$\Rightarrow$ such C has width $\geq \delta\sigma/\ell$ $\square$

Progress Measure Approach (2/4)

Given (s, δ, E)-semirespectful expander $(\mathcal{U}, \mathcal{V})_E$ for $\mathcal{F}$, define

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   - Fix minimal $\mathcal{U}_2$ s.t. $\bigwedge_{F \in \mathcal{U}_2} F \wedge E \vDash D \vee \overline{x}$
   - Then it holds that

$$\bigwedge_{F \in \mathcal{U}_1 \cup \mathcal{U}_2} F \wedge E \vDash C \vee D \ ,$$

so $\mu(C \vee D) \leq |\mathcal{U}_1 \cup \mathcal{U}_2| \leq |\mathcal{U}_1| + |\mathcal{U}_2| = \mu(C \vee x) + \mu(D \vee \overline{x})$

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   - So $\bigwedge_{F_i \in \mathcal{U}'} F_i \land E \nvDash \bot$ for $|\mathcal{U}'| \leq s$ and hence $\mu(\bot) > s$

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Given (s, δ, E)-semirespectful expander $(\mathcal{U}, \mathcal{V})_E$ with overlap ℓ

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Proof of claim: Another flipping argument using semirespectfulness

- Fix $V \in \partial^{\mathsf{sr}}_E(\mathcal{U}_C)$ and unique neighbour $F_V \in \mathcal{U}_C$ of V
- By minimality, $\exists\, \alpha$ s.t. $\alpha\big(\bigwedge_{F \in \mathcal{U}_C \setminus \{F_V\}} F \wedge E\big) = 1$ but $\alpha(C) = 0$
- If $V \cap \mathit{Vars}(C) = \emptyset$, then E-semirespectfully flip α on V to satisfy F_V $\lightning$

Applications: Tseitin and Onto-FPHP

Tseitin formulas

- F_i = clauses encoding parity constraint for ith vertex
- V_j = singleton set with jth edge (so overlap $\ell = 1$)
- $E = \emptyset$
- If underlying graph edge expander, then $(\mathcal{U}, \mathcal{V})_E$-graph semirespectful boundary expander with same parameters

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Onto functional PHP formulas

- F_i = singleton set with pigeon axiom for pigeon i
- V_j = all variables $p_{i,j}$ mentioning hole j (again overlap $\ell = 1$)
- E = all hole, functional, and onto axioms
- If onto FPHP restricted to bipartite graph, then $(\mathcal{U}, \mathcal{V})_E$-graph semirespectful boundary expander with same parameters

From Resolution to Polynomial Calculus

Obtain resolution width lower bounds from expander graphs where we can win following game on edges

Resolution edge game on (F, V) with side constraints E

1. Adversary provides total assignment α such that $\alpha(E) = 1$
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But Tseitin and onto FPHP both easy for polynomial calculus!

So semirespectful boundary expanders cannot yield any lower bounds for polynomial calculus

A Harder Edge Game for Polynomial Calculus

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To get polynomial calculus degree lower bounds need winning strategy for harder game on expander graphs

Polynomial calculus edge game on (F, V) with side constraints E

1. Commit to $\alpha_V : V \to \{0, 1\}$
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3. Flipping α on V to α_V should yield $\alpha[\alpha_V/V](F \wedge E) = 1$

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Not E-respectful — α_V needs $y \mapsto 0$, but $F_2{\restriction}_{y=0} = \{x \vee \overline{z},\, \overline{x} \vee z\}$

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Respectful Expanders and Degree Lower Bounds

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Respectful expander

An $(\mathcal{U}, \mathcal{V})_E$-graph is an (s, δ, E)-respectful expander if for all $\mathcal{U}' \subseteq \mathcal{U}$, $|\mathcal{U}'| \leq s$ it holds that $|\partial^{\mathsf{r}}_E(\mathcal{U}')| \geq \delta |\mathcal{U}'|$

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Theorem ([Mikša & Nordström '15] building on [AR01])

If $\mathcal{F}$ has (s, δ, E)-respectful expander $(\mathcal{U}, \mathcal{V})_E$ with overlap ℓ, then

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(Also holds for sets of polynomials not obtained from CNFs)

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Provides common framework for previous lower bounds:

- CNFs with expanding CVIGs [Alekhnovich & Razborov '01]
- "Vanilla" PHP formulas [Alekhnovich & Razborov '01]
- Ordering principle [Galesi & Lauria '10]
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New contribution: Functional PHP is hard

Hardness of Different Flavours of PHP

Variant	Resolution	Polynomial calculus
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- Prove that FPHP is hard in polynomial calculus

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If G is a (standard) bipartite (s, δ)-boundary expander with left degree $\leq d$, then $FPHP_G$ requires PC degree $> \delta s/(2d)$.

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- So get same expansion parameters, and theorem follows $\square$

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- Go beyond polynomial calculus (e.g. to Positivstellensatz, a.k.a. Lasserre/sums-of-squares)

Take-away Message

Generalized method for PC degree lower bounds

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Thank you for your attention!