Nullstellensatz Size-Degree Trade-offs from Reversible Pebbling

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AC Section lunch meeting

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Joint work with Susanna F. de Rezende, Or Meir, and Robert Robere

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- Degree: maximum degree (3 in example)
- Size: # monomials when expanded (7 in example)

Questions of interest

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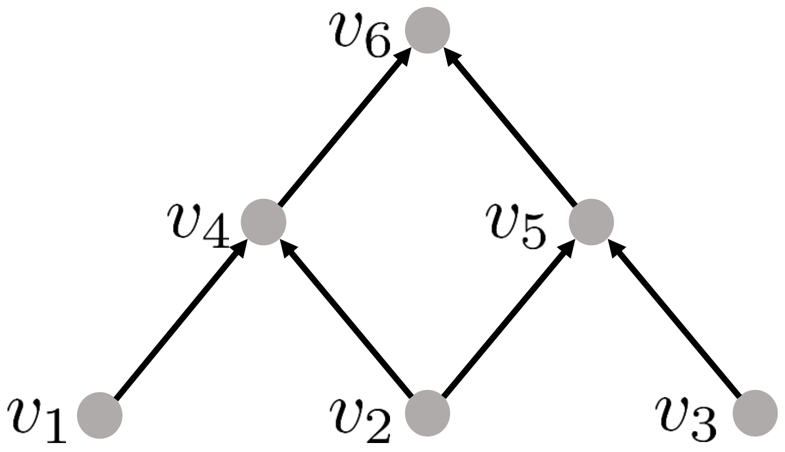
- Upper and lower bounds on size
- Size-degree relations? Simultaneous optimization?

Reversible pebble game



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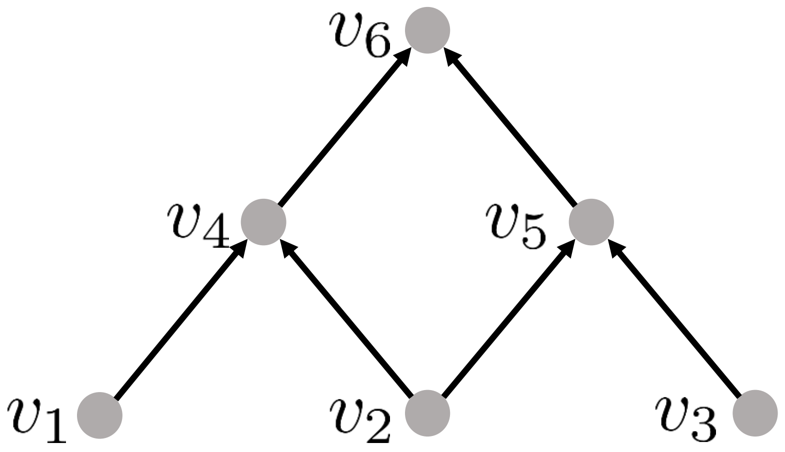

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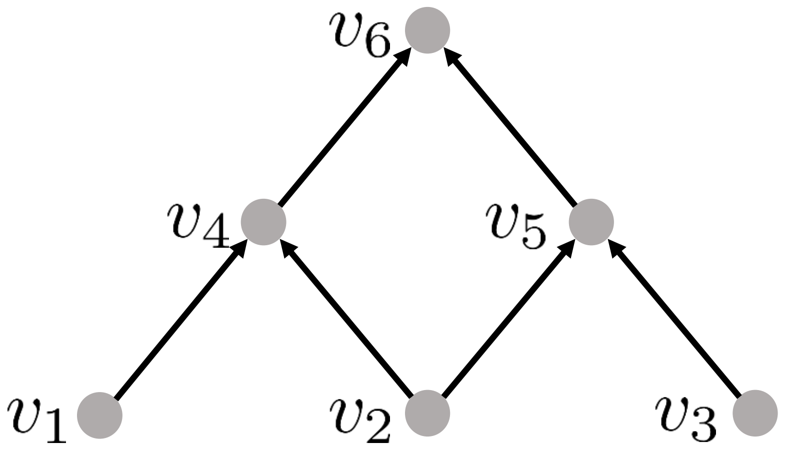

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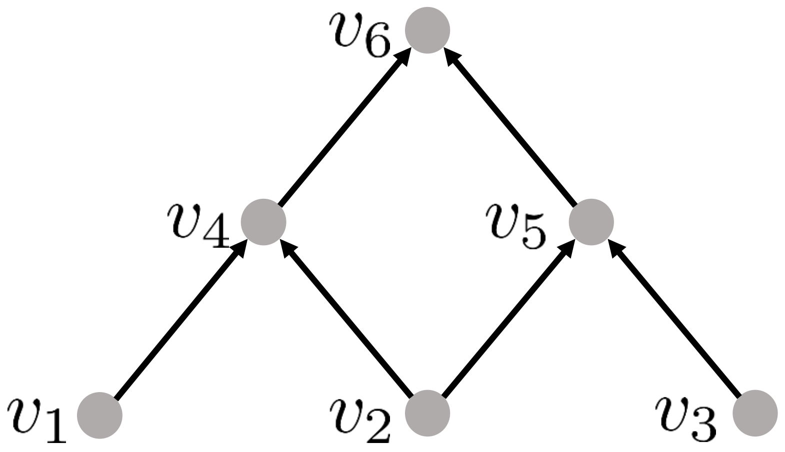

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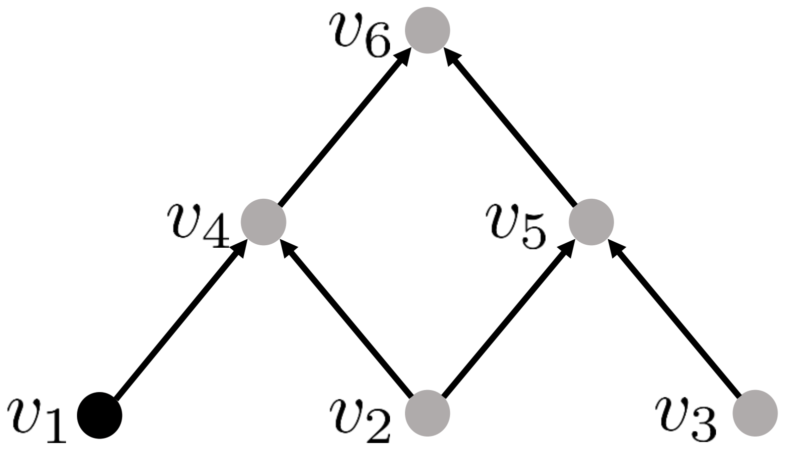

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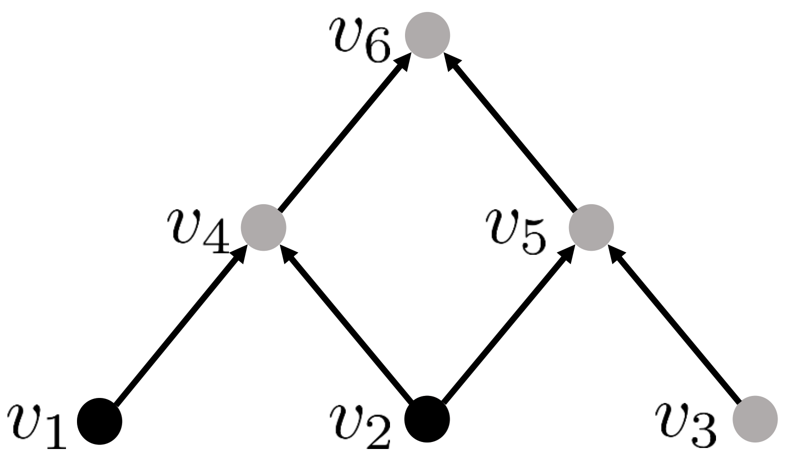

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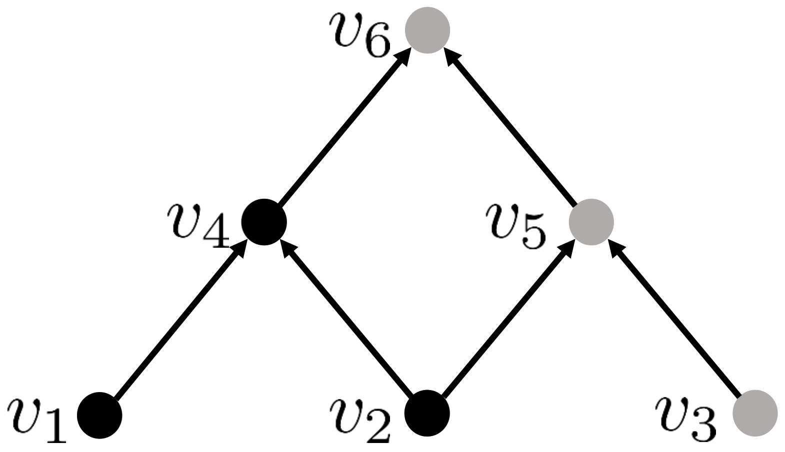

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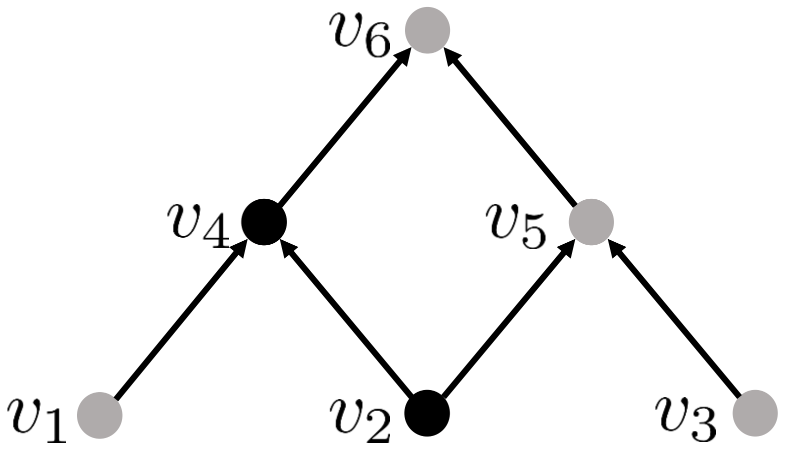

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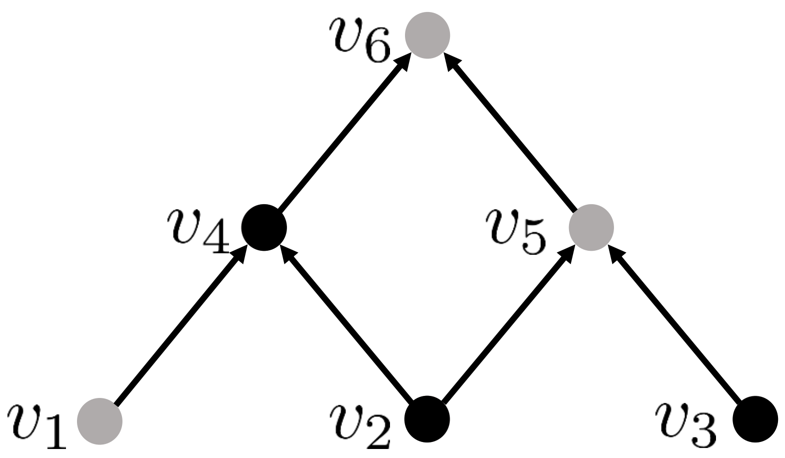

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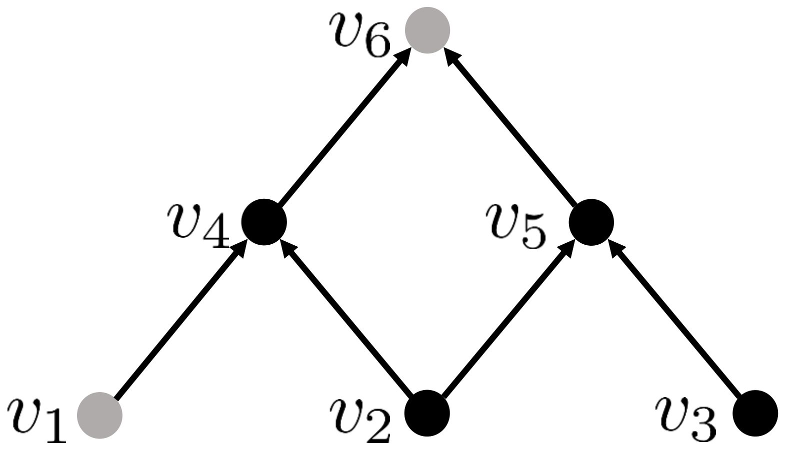

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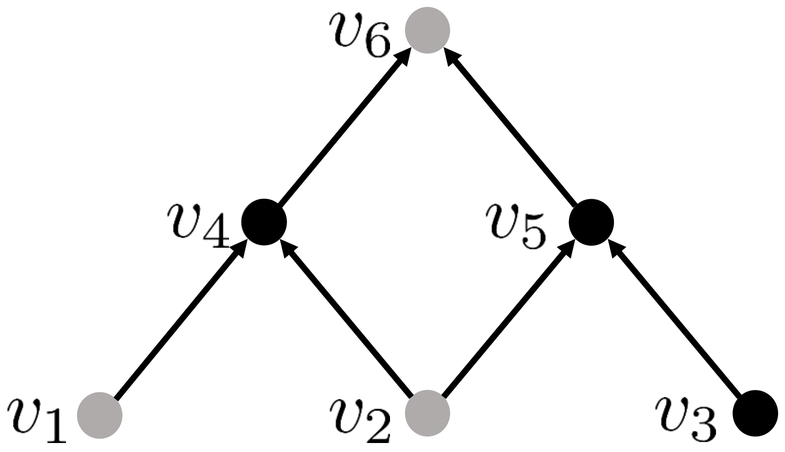

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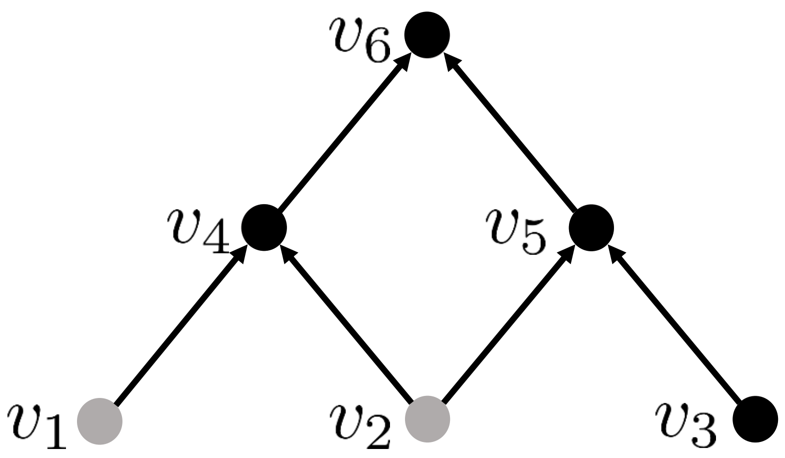

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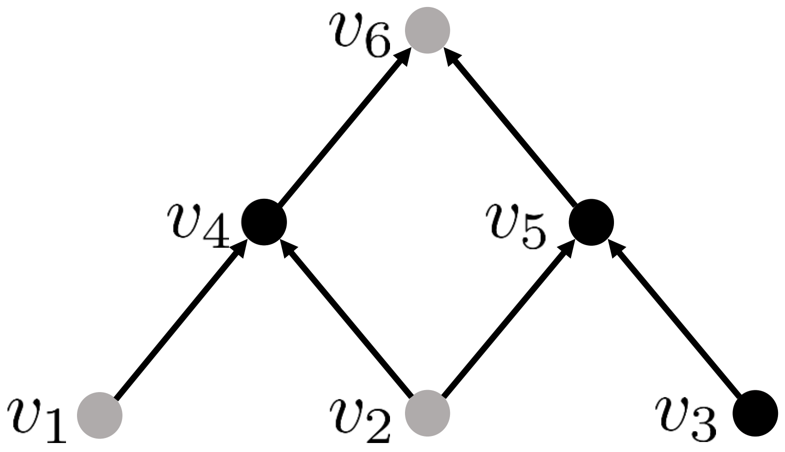

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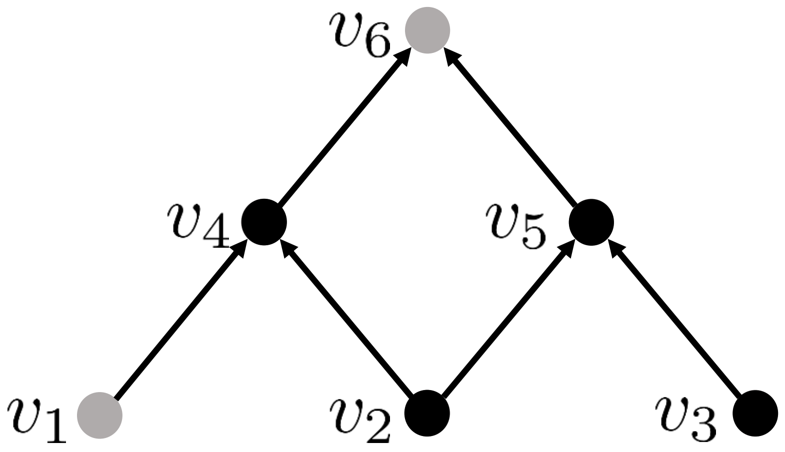

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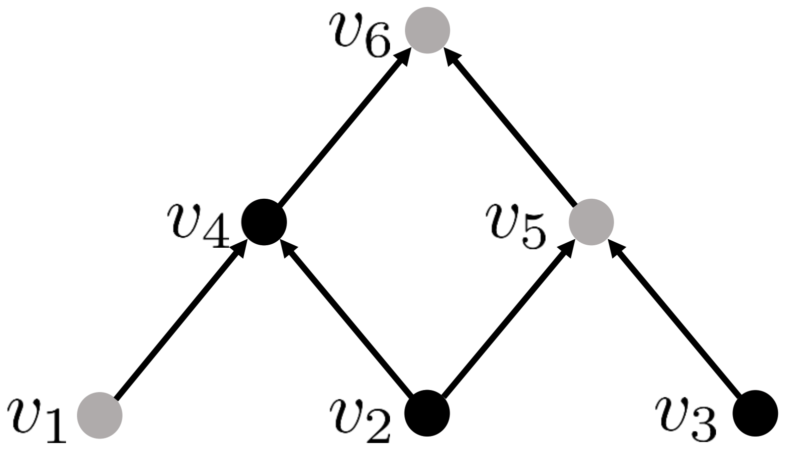

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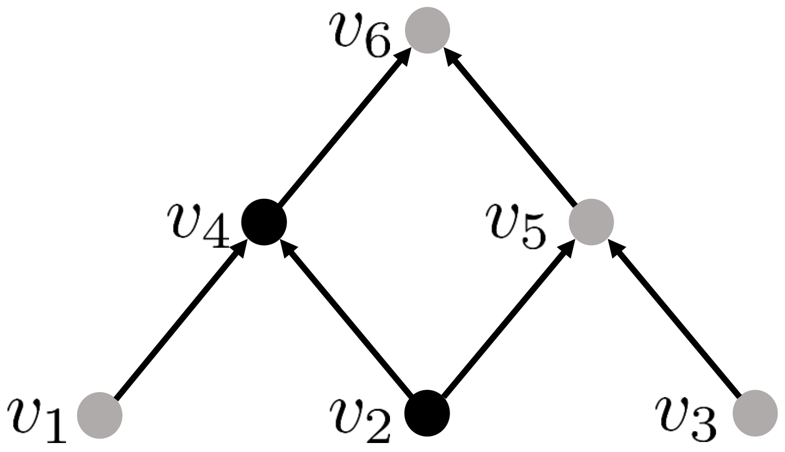

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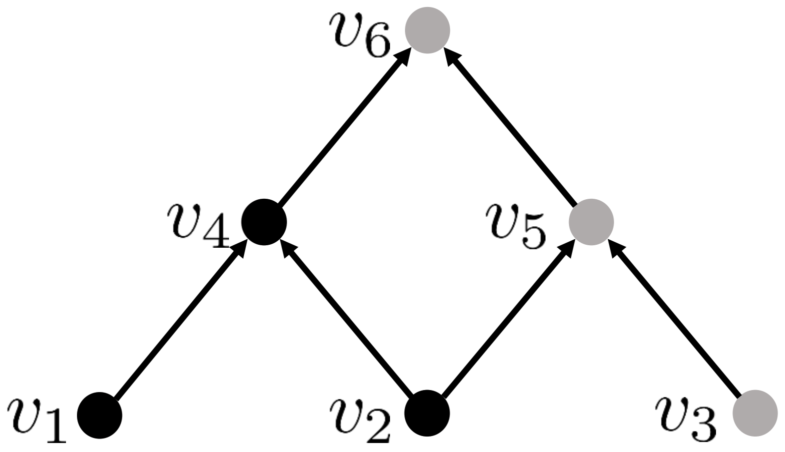

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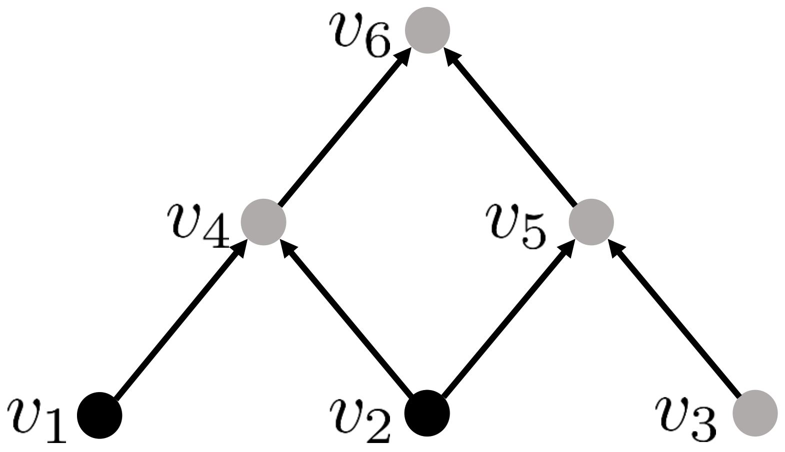

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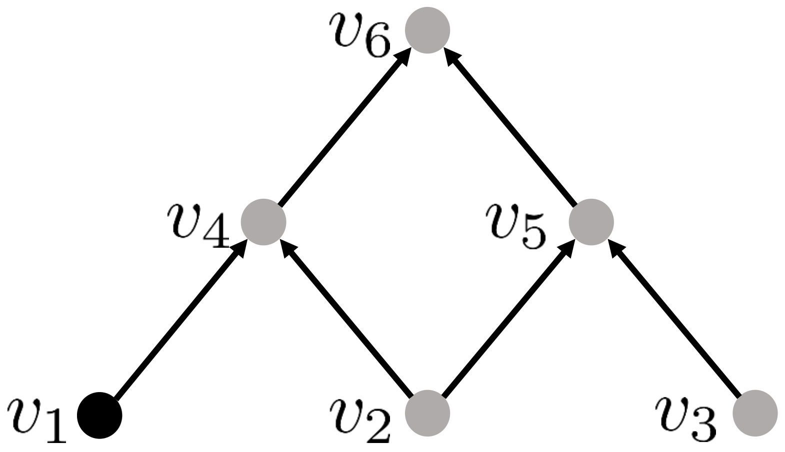

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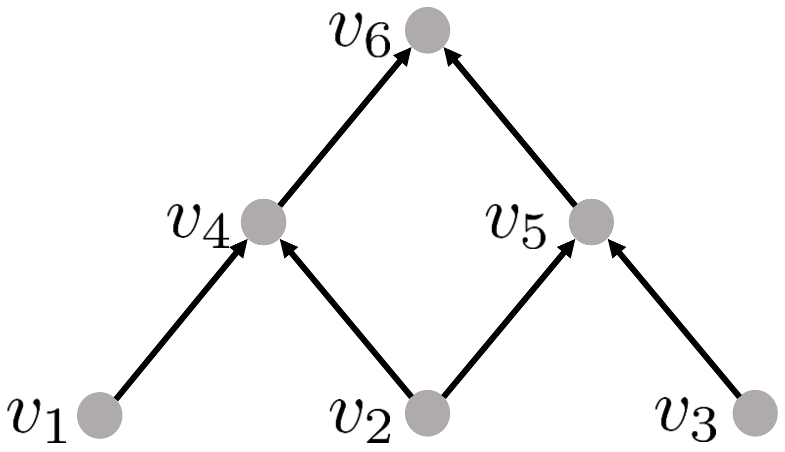

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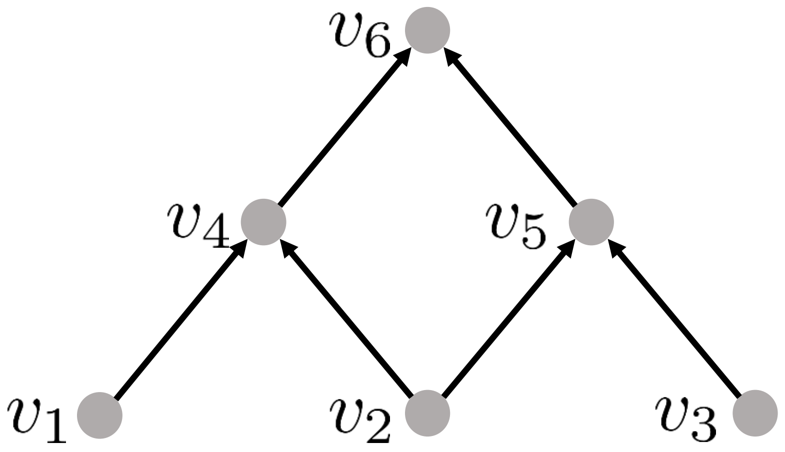

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- Space: maximum # pebbles in any configuration

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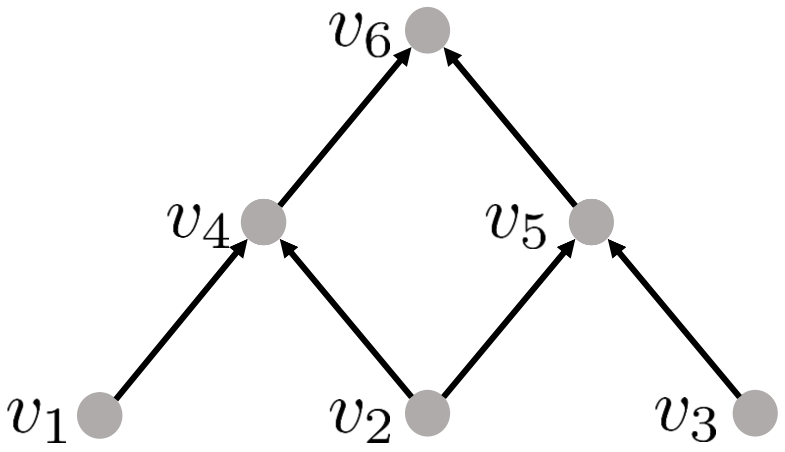

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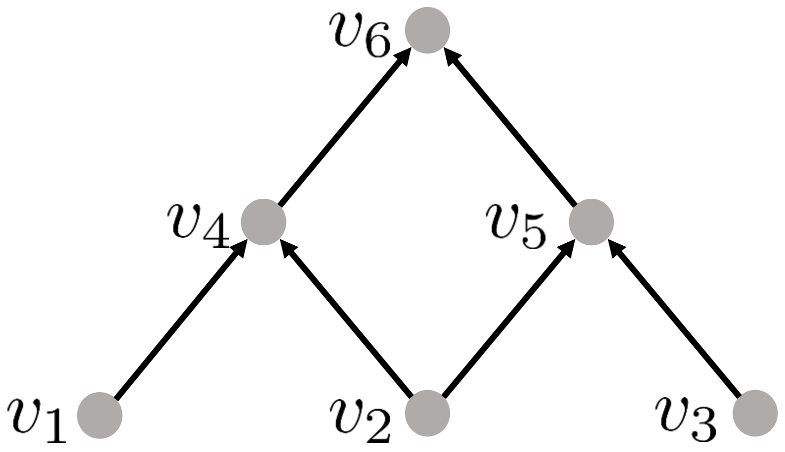

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- Time: # moves

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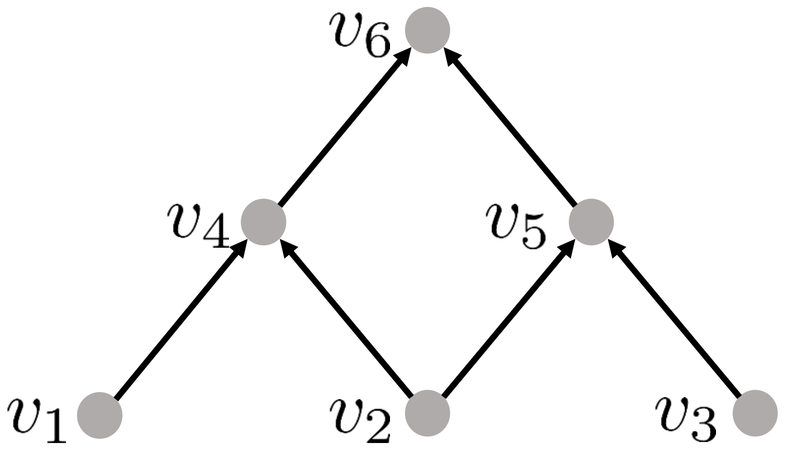

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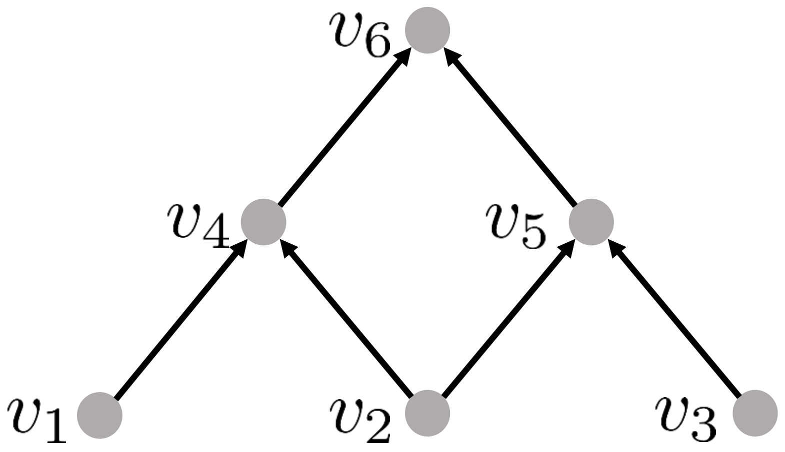
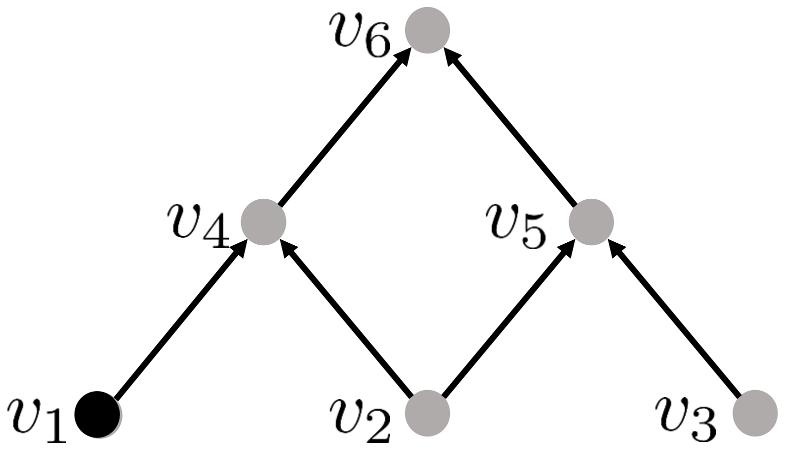
- Time: # moves $(t = 16$ in example)

Reversible pebble game



- Can do space 4, time 16
- Faster pebbling?

Reversible pebble game



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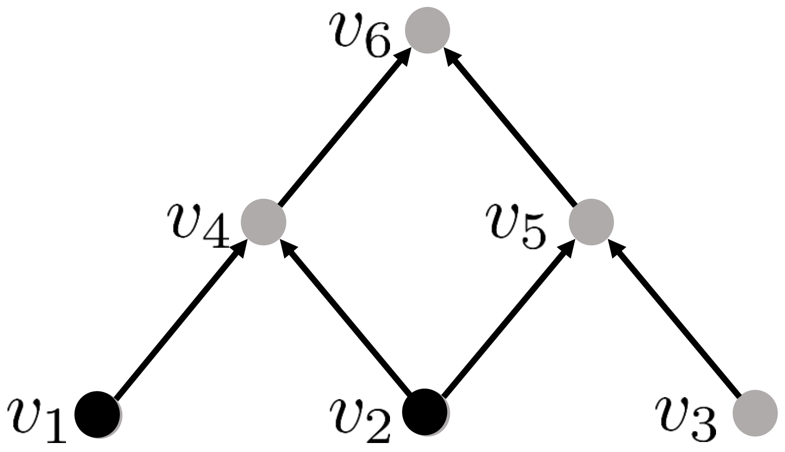

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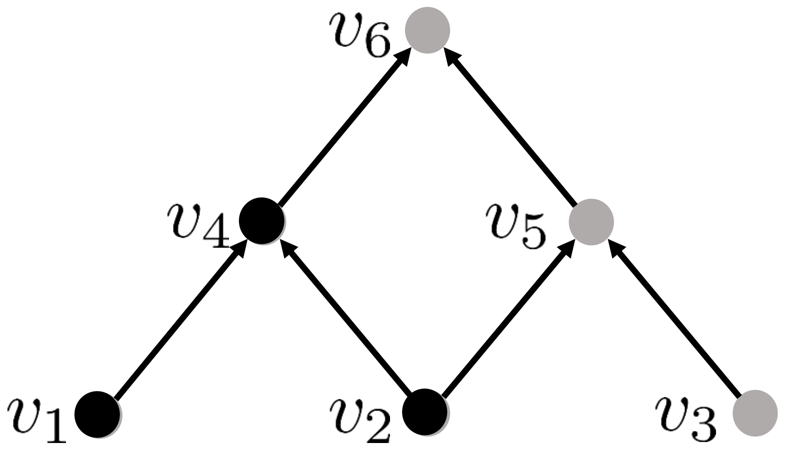

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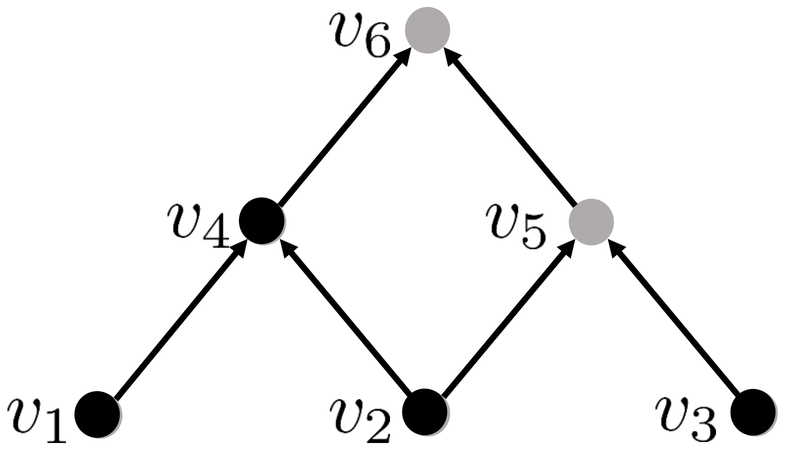

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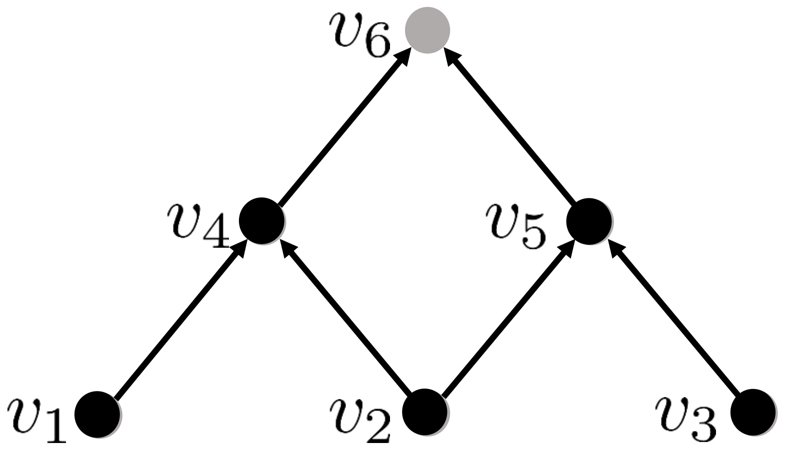

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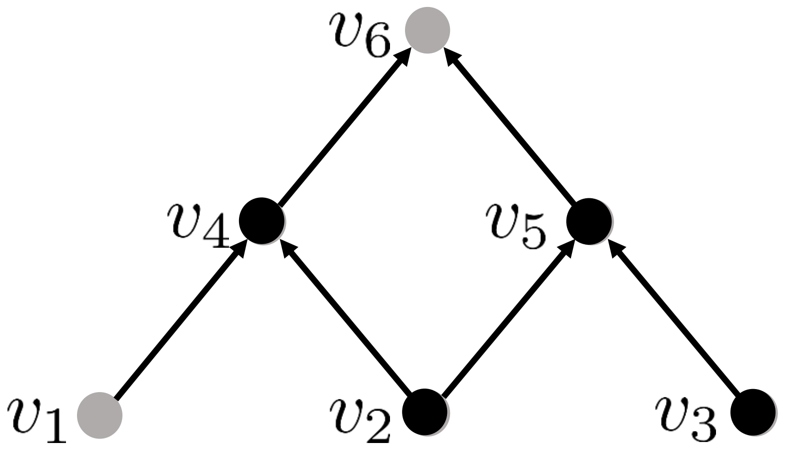

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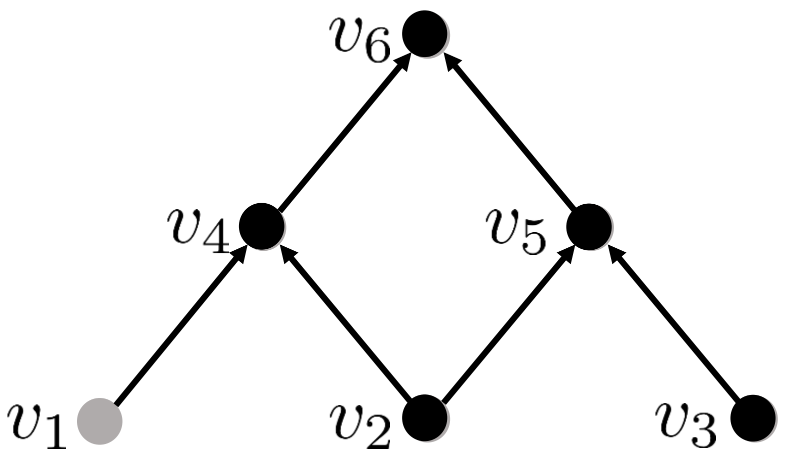

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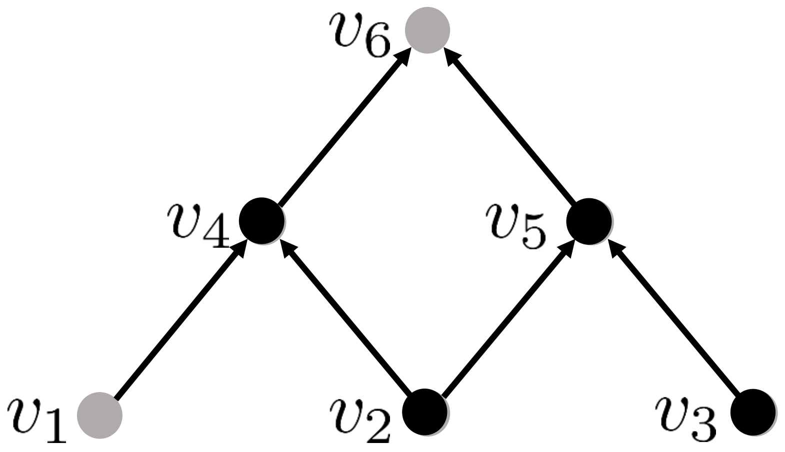

- Can do space 4, time 16
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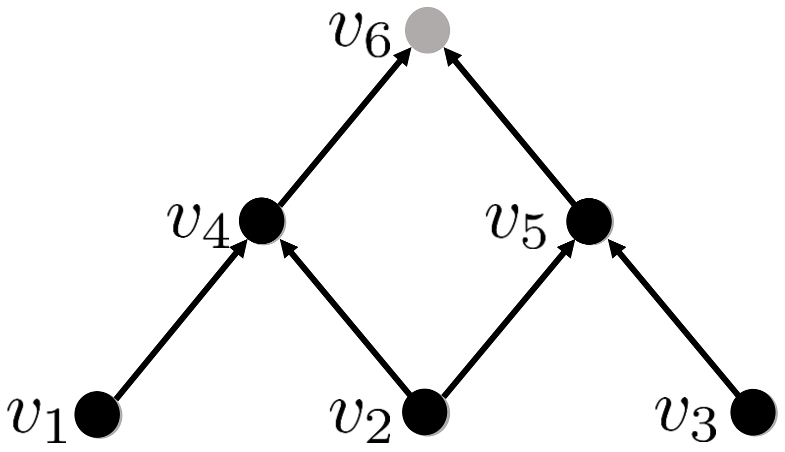

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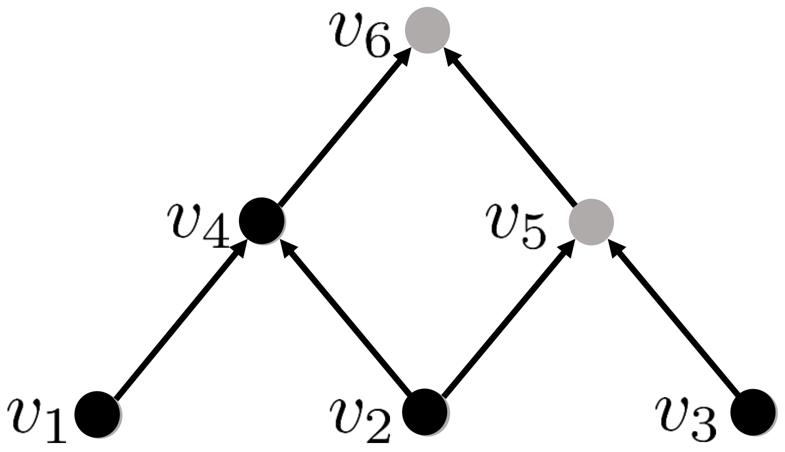

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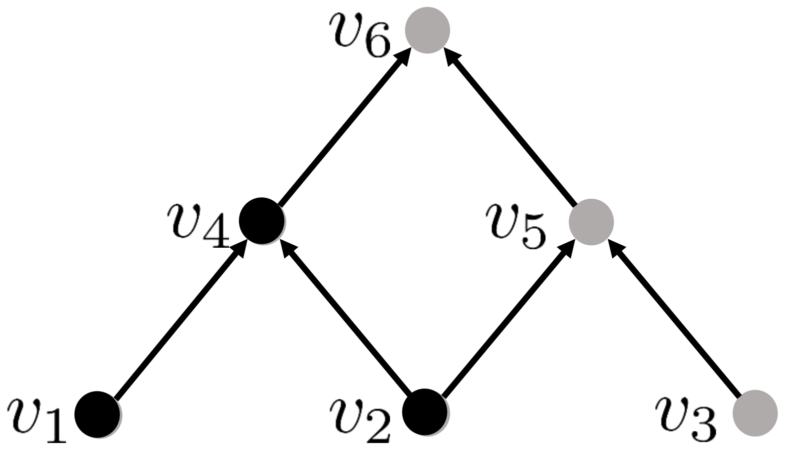

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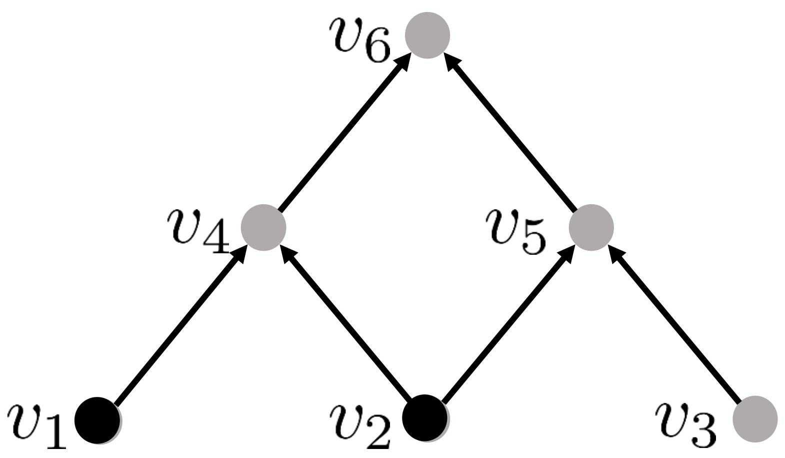

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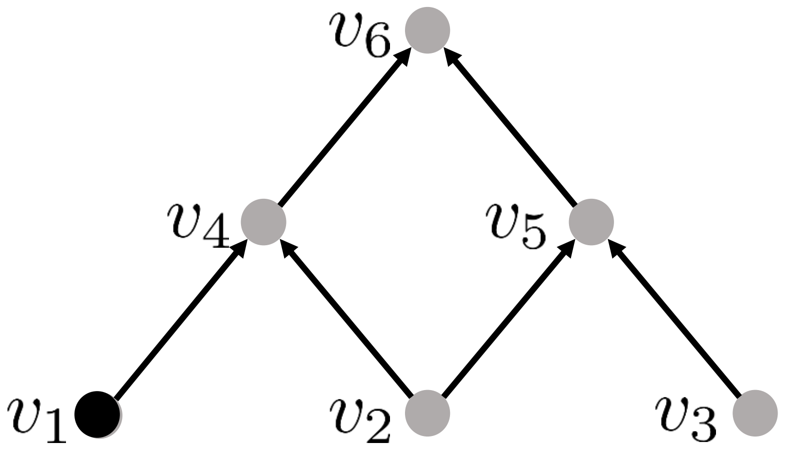

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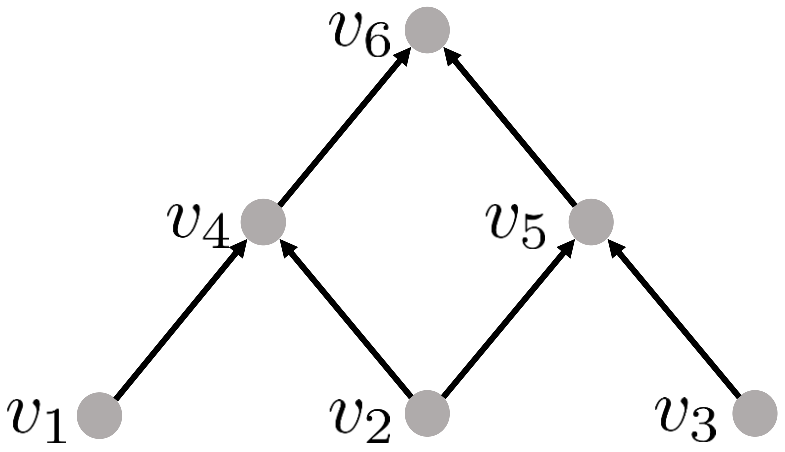

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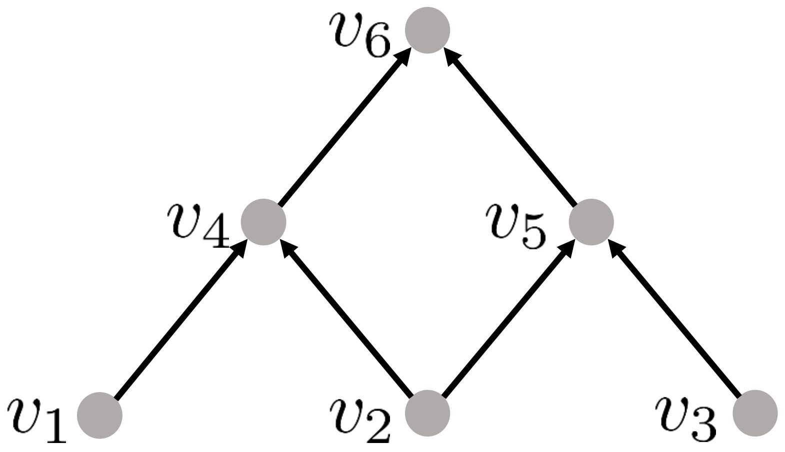

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space	time
4	16

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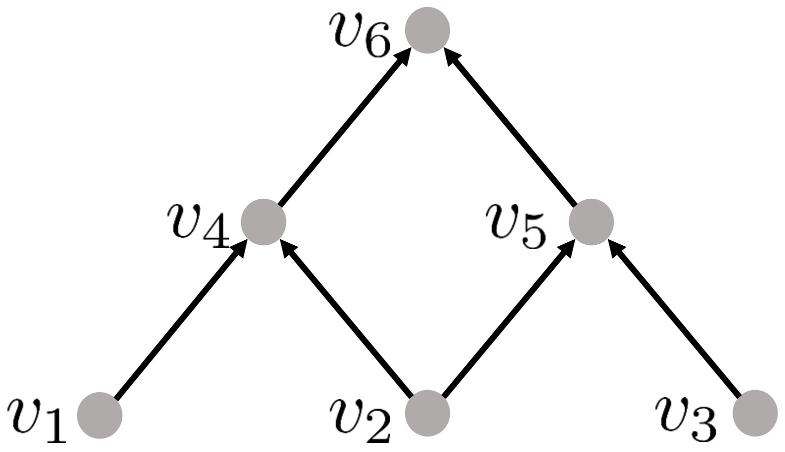

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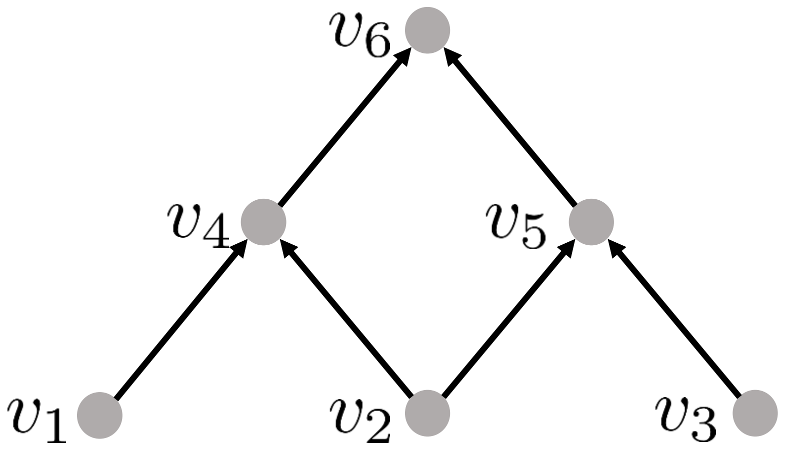
Reversible pebble game



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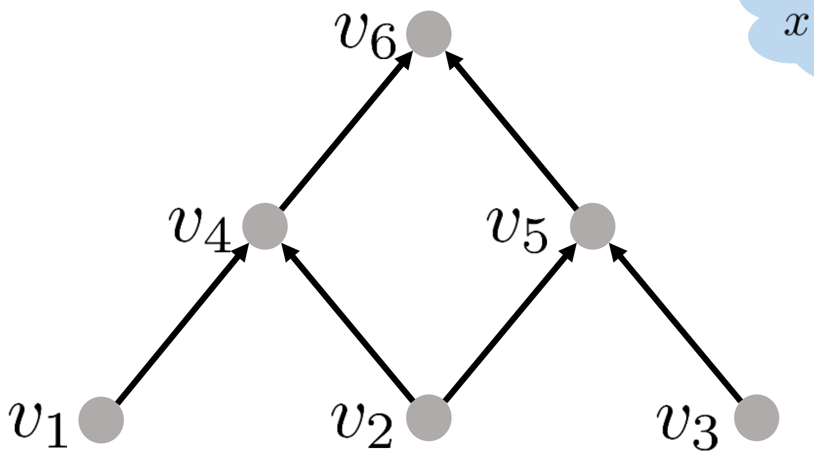
space	time
4	16
5	14
6	12

Pebbling contradiction Peb_G



x true $\Leftrightarrow x = 1$
x false $\Leftrightarrow x = 0$

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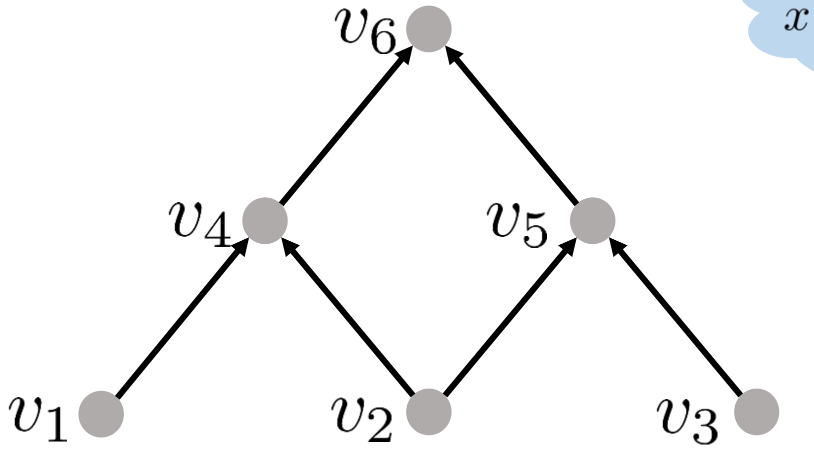

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$$A_v := (1 - x_v) \prod_{u \in \mathsf{pred}(v)} x_u, \quad \text{for all } v \in V(G)$$

$$A_{\mathsf{sink}} := x_{\mathsf{sink}}$$

$\mathsf{pred}(v)$: set of all predecessors of v

Nullstellensatz refutation

- Pebbling contradiction Peb_G

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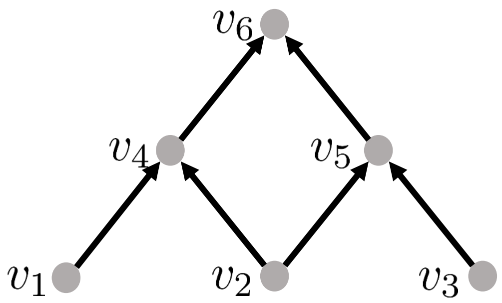
- NS refutation

$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$

Reversible pebbling

- DAG G



- Reversible pebbling

$$\emptyset = \mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_t = \emptyset$$

Nullstellensatz refutation

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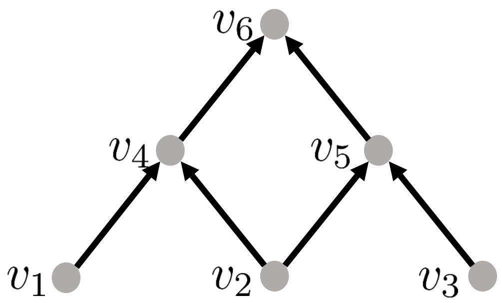

- Reversible pebbling

$$\emptyset = \mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_t = \emptyset$$

Theorem

$\exists$ NS refutation of Peb_G in size $t + 1$ and degree s

$\Leftrightarrow$

$\exists$ reversible pebbling of G in time t and space s

Resolution

- Reason with clauses

$$\frac{y \lor \neg x \qquad z \lor y \lor x}{z \lor y}$$

Resolution

- Reason with clauses
- Measure size, width

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Polynomial calculus

- Reason with polynomials

$$\frac{2xy + 3xz = 0 \qquad xyz - xz = 0}{}$$

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$$\frac{2xy + 3xz = 0 \qquad 3 \cdot (xyz - xz = 0)}{2xy + 3xyz = 0}$$

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- Small degree/width $\Rightarrow$ small size

  Only $\binom{2n}{\leq w} \leq (2n)^w$ clauses of width $\leq w$ (essentially tight [ALN16])

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- Small size $\Rightarrow$ (medium-)small degree/width [IPS99, BW01]

Size-degree relation

Polynomial calculus, resolution

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Size-degree relation

Polynomial calculus, resolution

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Nullstellensatz

- Small degree $\Rightarrow$ small size
- Small size $\not\Rightarrow$ small degree

Size-degree relation

Polynomial calculus, resolution

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- Small size $\Rightarrow$ (medium-)small degree/width [IPS99, BW01]

Nullstellensatz

- Small degree $\Rightarrow$ small size
- Small size $\not\Rightarrow$ small degree

Small size $\Rightarrow$ small degree: reduction blows up size. Inherent?

- Resolution: yes, strong size-width trade-offs [Tha16]
- Polynomial calculus: open
- Nullstellensatz: strong size-degree trade-offs [this work]

Nullstellensatz size-degree trade-offs

Theorem

There is an explicit family of sets of polynomials s.t.

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Nullstellensatz size-degree trade-offs

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Nullstellensatz size-degree trade-offs

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Nullstellensatz size-degree trade-offs

Theorem

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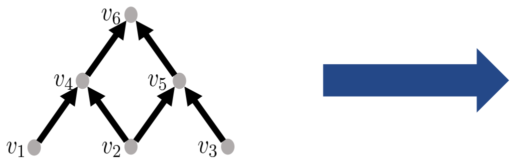
Proof.

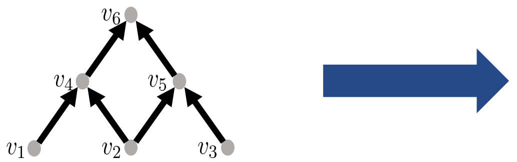
- $\exists$ NS refutation in size $t + 1$, degree $s \Leftrightarrow \exists$ reversible pebbling in time t, space s
- Show strong reversible pebbling time-space trade-offs

Reversible pebbling to NS refutation



$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$

Reversible pebbling to NS refutation



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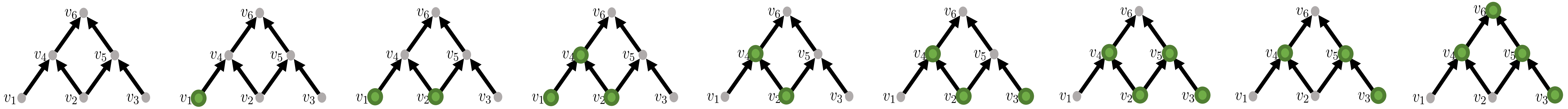

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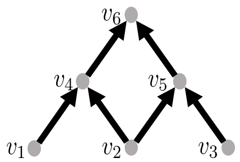

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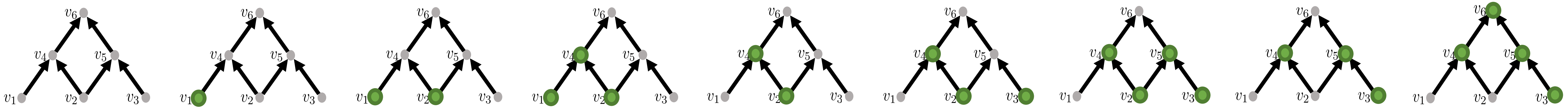

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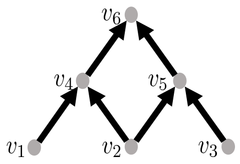

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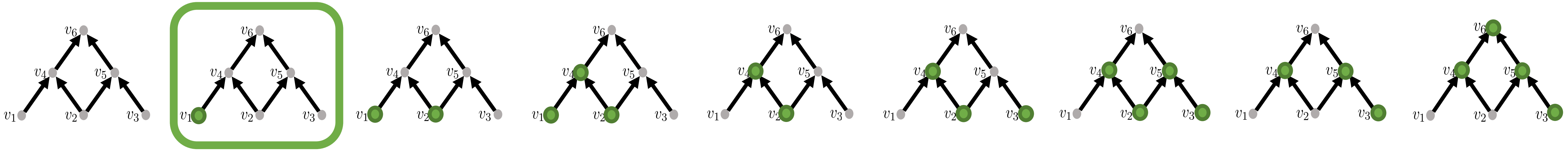

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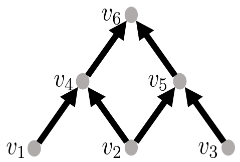

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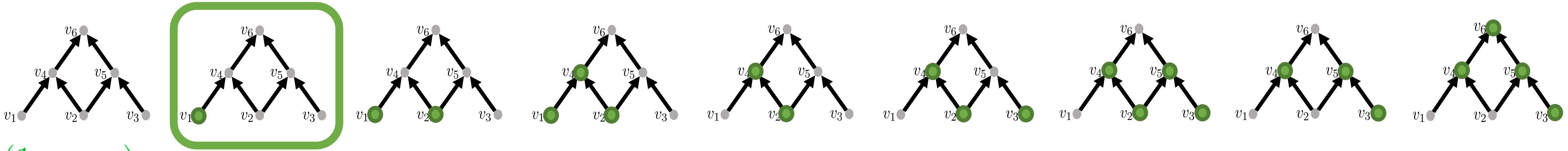

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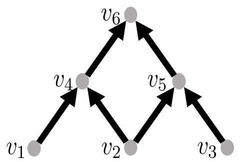

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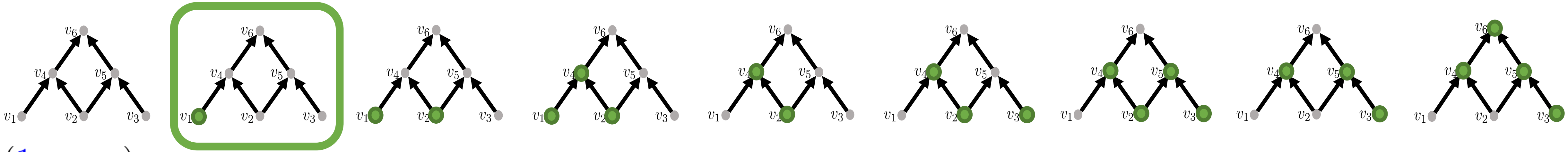

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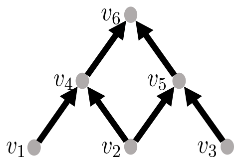

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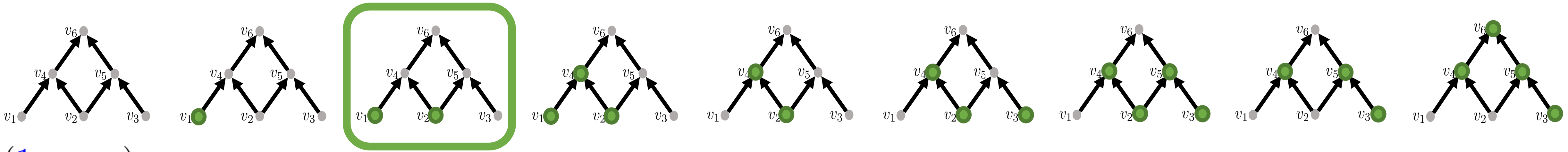

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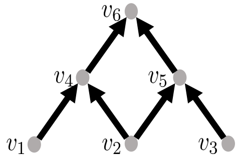

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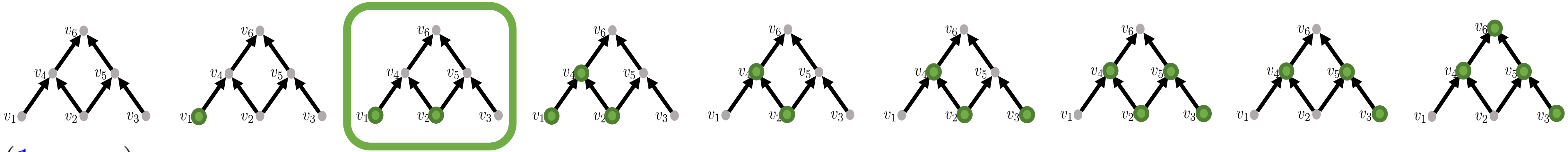

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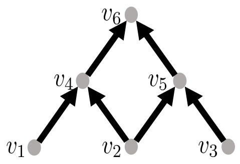

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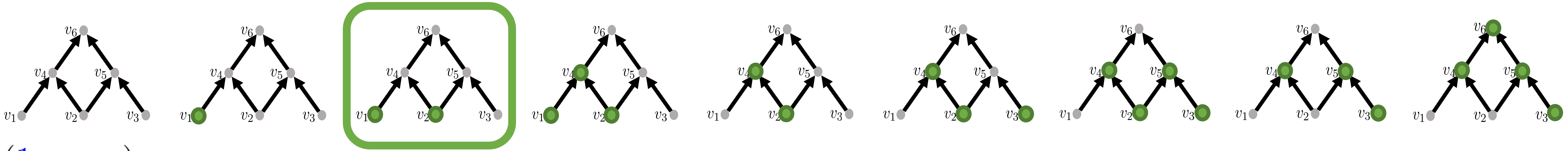

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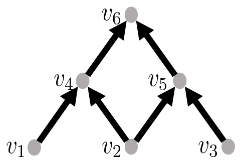

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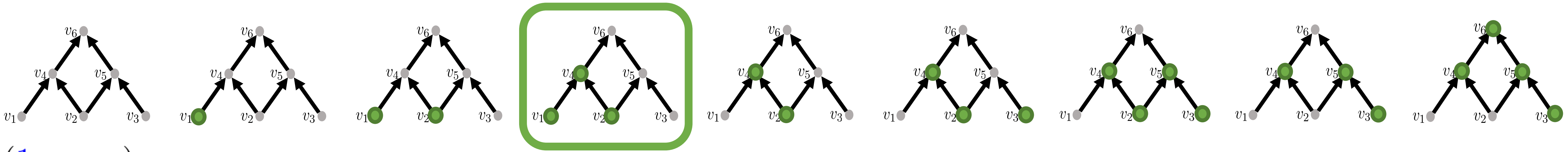

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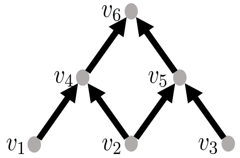

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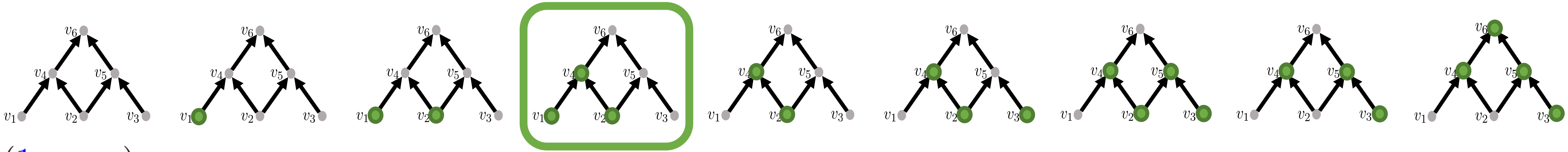

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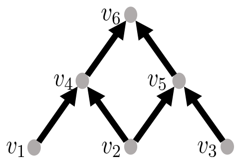
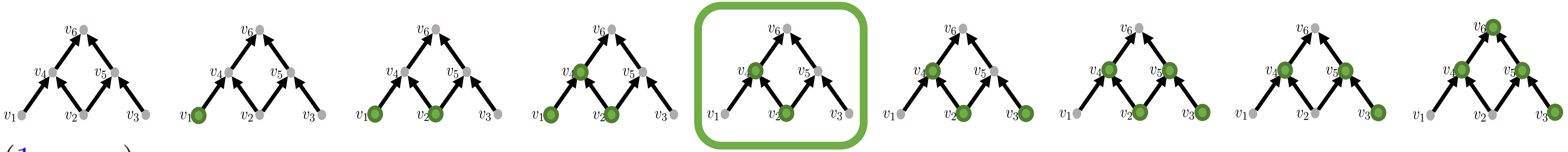
Reversible pebbling to NS refutation



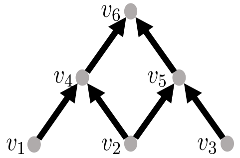
$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



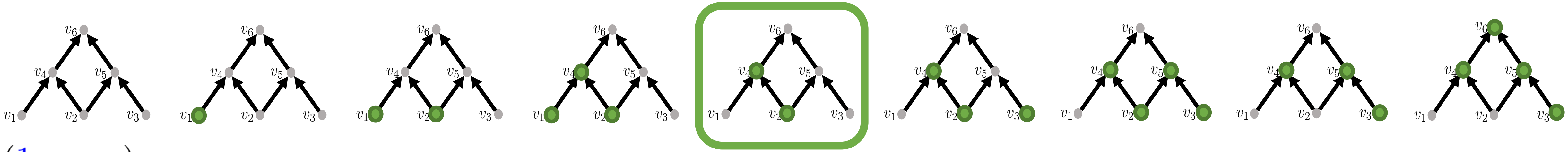
$$(1 - x_{v_1})$$
$$+ x_{v_1}(1 - x_{v_2})$$
$$+ x_{v_1}x_{v_2}(1 - x_{v_4})$$
$$+ x_{v_2}x_{v_4}(-1)(1 - x_{v_1})$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$
$$x_{v_2}x_{v_3}(1 - x_{v_5}) = 0$$
$$x_{v_4}x_{v_5}(1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



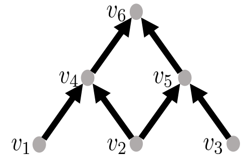
$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



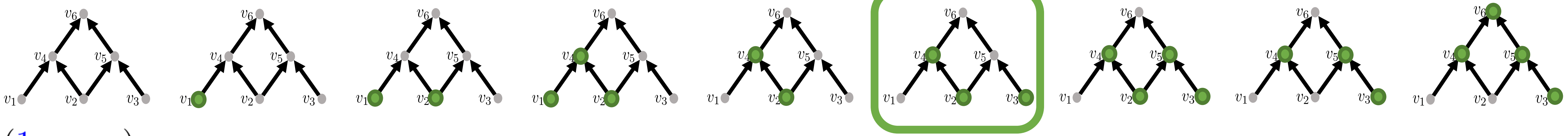
$$(1 - x_{v_1}) + x_{v_1}(1 - x_{v_2}) + x_{v_1}x_{v_2}(1 - x_{v_4}) + x_{v_2}x_{v_4}(-1)(1 - x_{v_1})$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$
$$x_{v_2}x_{v_3}(1 - x_{v_5}) = 0$$
$$x_{v_4}x_{v_5}(1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



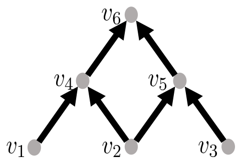
$$\sum_{v} Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



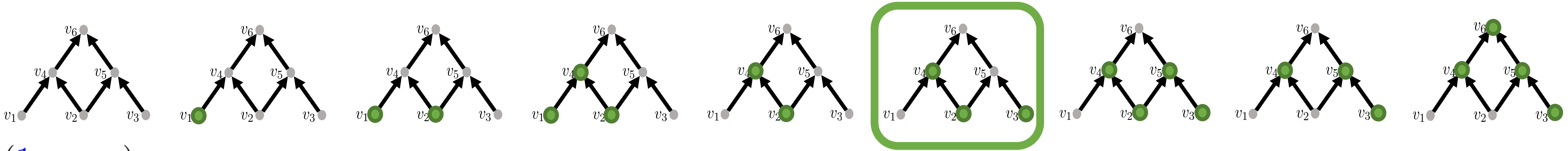
$$(1 - x_{v_1})$$
$$+ x_{v_1}(1 - x_{v_2})$$
$$+ x_{v_1} x_{v_2}(1 - x_{v_4})$$
$$+ x_{v_2} x_{v_4}(-1)(1 - x_{v_1})$$
$$+ x_{v_2} x_{v_4}(1 - x_{v_3})$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1} x_{v_2}(1 - x_{v_4}) = 0$$
$$x_{v_2} x_{v_3}(1 - x_{v_5}) = 0$$
$$x_{v_4} x_{v_5}(1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



$$(1 - x_{v_1}) + x_{v_1}(1 - x_{v_2}) + x_{v_1}x_{v_2}(1 - x_{v_4}) + x_{v_2}x_{v_4}(-1)(1 - x_{v_1}) + x_{v_2}x_{v_4}(1 - x_{v_3})$$

$$1 - x_{v_1} = 0$$

$$1 - x_{v_2} = 0$$

$$1 - x_{v_3} = 0$$

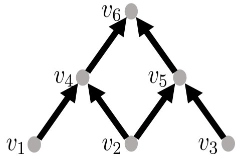
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$

$$x_{v_2}x_{v_3}(1 - x_{v_5}) = 0$$

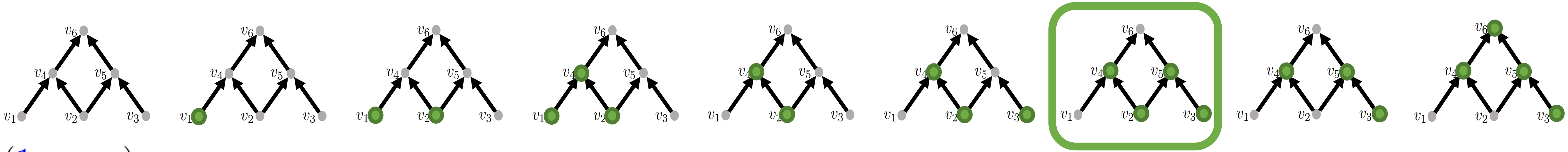
$$x_{v_4}x_{v_5}(1 - x_{v_6}) = 0$$

$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



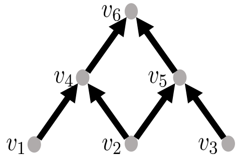
$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



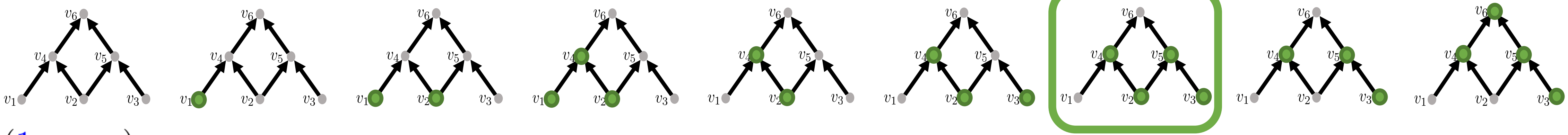
$$
\begin{aligned}
&(1 - x_{v_1}) \\
&\quad + x_{v_1}(1 - x_{v_2}) \\
&\qquad + x_{v_1}x_{v_2}(1 - x_{v_4}) \\
&\qquad\quad + x_{v_2}x_{v_4}(-1)(1 - x_{v_1}) \\
&\qquad\qquad + x_{v_2}x_{v_4}(1 - x_{v_3}) \\
&\qquad\qquad\quad + x_{v_2}x_{v_3}x_{v_4}(1 - x_{v_5})
\end{aligned}
$$

$$
\begin{aligned}
1 - x_{v_1} &= 0 \\
1 - x_{v_2} &= 0 \\
1 - x_{v_3} &= 0 \\
x_{v_1}x_{v_2}(1 - x_{v_4}) &= 0 \\
x_{v_2}x_{v_3}(1 - x_{v_5}) &= 0 \\
x_{v_4}x_{v_5}(1 - x_{v_6}) &= 0 \\
x_{v_6} &= 0
\end{aligned}
$$

Reversible pebbling to NS refutation



$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



$$
\begin{aligned}
&(1 - x_{v_1}) \\
&\quad + x_{v_1}(1 - x_{v_2}) \\
&\qquad + x_{v_1}x_{v_2}(1 - x_{v_4}) \\
&\qquad\quad + x_{v_2}x_{v_4}(-1)(1 - x_{v_1}) \\
&\qquad\qquad + x_{v_2}x_{v_4}(1 - x_{v_3}) \\
&\qquad\qquad\quad + x_{v_2}x_{v_3}x_{v_4}(1 - x_{v_5})
\end{aligned}
$$

$$1 - x_{v_1} = 0$$

$$1 - x_{v_2} = 0$$

$$1 - x_{v_3} = 0$$

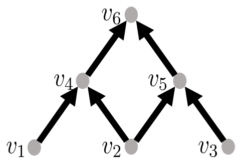
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$

$$x_{v_2}x_{v_3}(1 - x_{v_5}) = 0$$

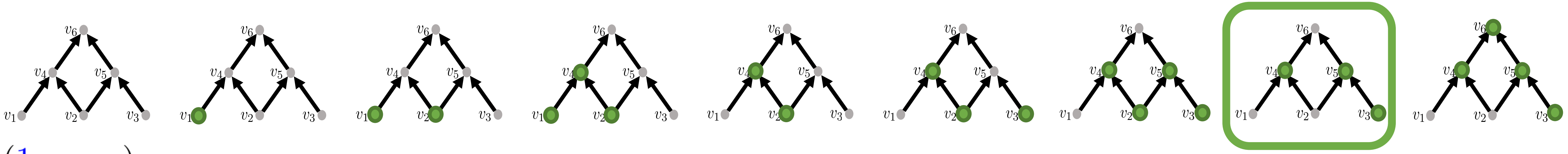
$$x_{v_4}x_{v_5}(1 - x_{v_6}) = 0$$

$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



$$(1 - x_{v_1}) + x_{v_1}(1 - x_{v_2}) + x_{v_1}x_{v_2}(1 - x_{v_4}) + x_{v_2}x_{v_4}(-1)(1 - x_{v_1}) + x_{v_2}x_{v_4}(1 - x_{v_3}) + x_{v_2}x_{v_3}x_{v_4}(1 - x_{v_5}) + x_{v_3}x_{v_4}x_{v_5}(-1)(1 - x_{v_2})$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$
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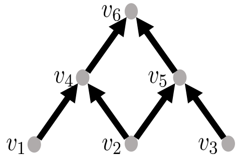
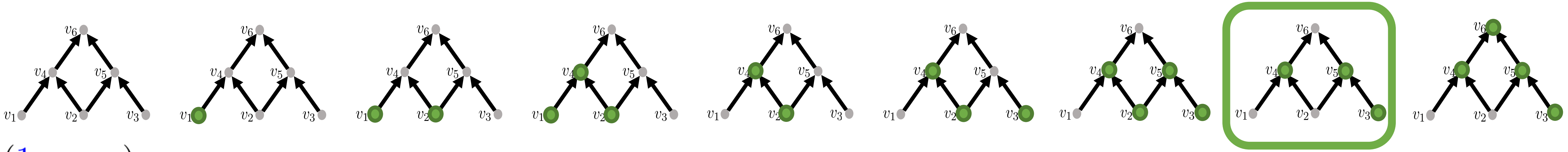
Reversible pebbling to NS refutation



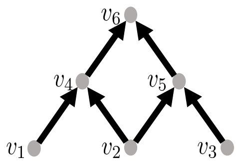
$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



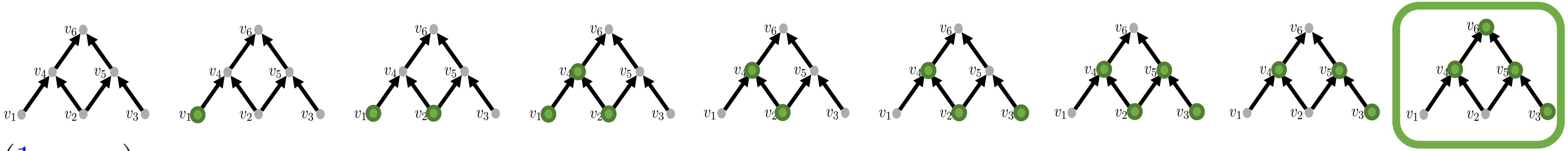
$$(1 - x_{v_1})$$
$$+ x_{v_1}(1 - x_{v_2})$$
$$+ x_{v_1}x_{v_2}(1 - x_{v_4})$$
$$+ x_{v_2}x_{v_4}(-1)(1 - x_{v_1})$$
$$+ x_{v_2}x_{v_4}(1 - x_{v_3})$$
$$+ x_{v_2}x_{v_3}x_{v_4}(1 - x_{v_5})$$
$$+ x_{v_3}x_{v_4}x_{v_5}(-1)(1 - x_{v_2})$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$
$$x_{v_2}x_{v_3}(1 - x_{v_5}) = 0$$
$$x_{v_4}x_{v_5}(1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



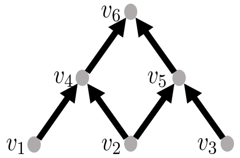
$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



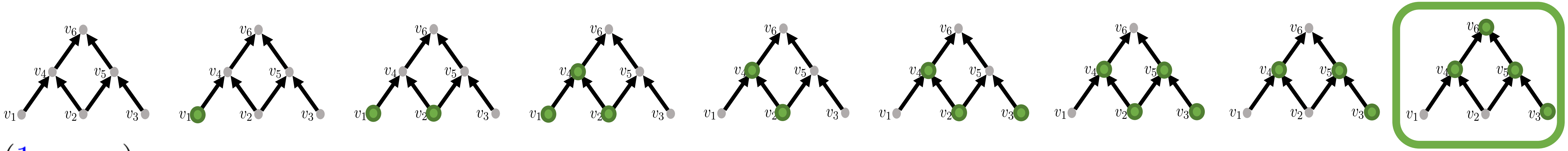
$$(1 - x_{v_1}) + x_{v_1}(1 - x_{v_2}) + x_{v_1}x_{v_2}(1 - x_{v_4}) + x_{v_2}x_{v_4}(-1)(1 - x_{v_1}) + x_{v_2}x_{v_4}(1 - x_{v_3}) + x_{v_2}x_{v_3}x_{v_4}(1 - x_{v_5}) + x_{v_3}x_{v_4}x_{v_5}(-1)(1 - x_{v_2}) + x_{v_3}x_{v_4}x_{v_5}(1 - x_{v_6})$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$
$$x_{v_2}x_{v_3}(1 - x_{v_5}) = 0$$
$$x_{v_4}x_{v_5}(1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



$$(1 - x_{v_1}) + x_{v_1}(1 - x_{v_2}) + x_{v_1}x_{v_2}(1 - x_{v_4}) + x_{v_2}x_{v_4}(-1)(1 - x_{v_1}) + x_{v_2}x_{v_4}(1 - x_{v_3}) + x_{v_2}x_{v_3}x_{v_4}(1 - x_{v_5}) + x_{v_3}x_{v_4}x_{v_5}(-1)(1 - x_{v_2}) + x_{v_3}x_{v_4}x_{v_5}(1 - x_{v_6})$$

$$1 - x_{v_1} = 0$$

$$1 - x_{v_2} = 0$$

$$1 - x_{v_3} = 0$$

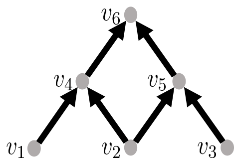
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$

$$x_{v_2}x_{v_3}(1 - x_{v_5}) = 0$$

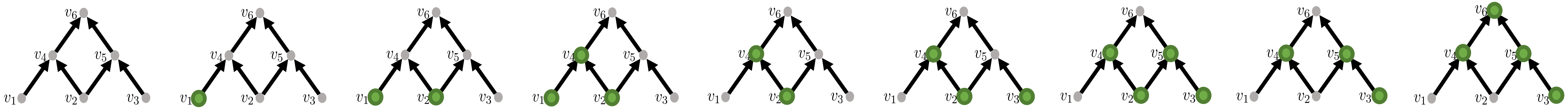
$$x_{v_4}x_{v_5}(1 - x_{v_6}) = 0$$

$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



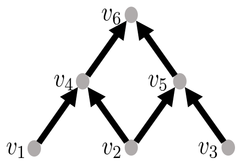
$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



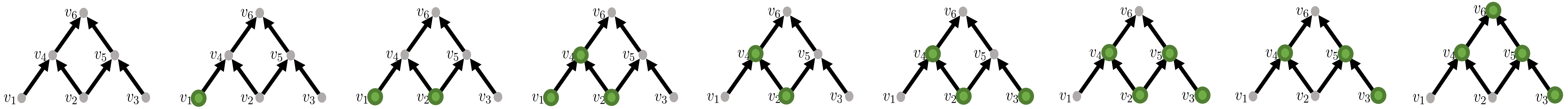
$$(1 - x_{v_1})$$
$$+ x_{v_1}(1 - x_{v_2})$$
$$+ x_{v_1}x_{v_2}(1 - x_{v_4})$$
$$+ x_{v_2}x_{v_4}(-1)(1 - x_{v_1})$$
$$+ x_{v_2}x_{v_4}(1 - x_{v_3})$$
$$+ x_{v_2}x_{v_3}x_{v_4}(1 - x_{v_5})$$
$$+ x_{v_3}x_{v_4}x_{v_5}(-1)(1 - x_{v_2})$$
$$+ x_{v_3}x_{v_4}x_{v_5}(1 - x_{v_6})$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$
$$x_{v_2}x_{v_3}(1 - x_{v_5}) = 0$$
$$x_{v_4}x_{v_5}(1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



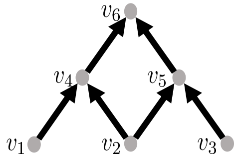
$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



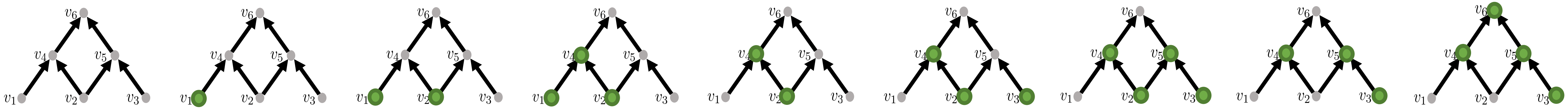
$$(1 - x_{v_1}) + (x_{v_1} - x_{v_1}x_{v_2}) + (x_{v_1}x_{v_2} - x_{v_1}x_{v_2}x_{v_4}) + (x_{v_1}x_{v_2}x_{v_4} - x_{v_2}x_{v_4}) + (x_{v_2}x_{v_4} - x_{v_2}x_{v_3}x_{v_4}) + (x_{v_2}x_{v_3}x_{v_4} - x_{v_2}x_{v_3}x_{v_4}x_{v_5}) + (x_{v_2}x_{v_3}x_{v_4}x_{v_5} - x_{v_3}x_{v_4}x_{v_5}) + (x_{v_3}x_{v_4}x_{v_5} - x_{v_3}x_{v_4}x_{v_5}x_{v_6})$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1}x_{v_2}(1 - x_{v_4}) = 0$$
$$x_{v_2}x_{v_3}(1 - x_{v_5}) = 0$$
$$x_{v_4}x_{v_5}(1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



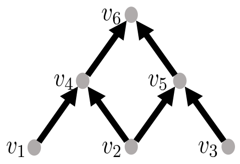
$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



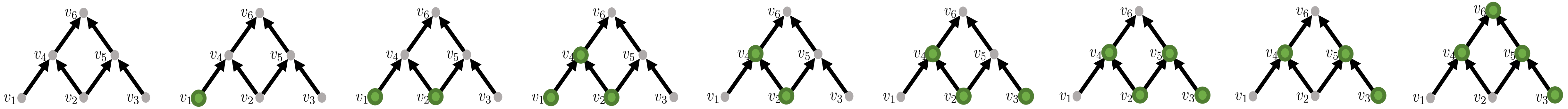
$$(1 - x_{v_1})$$
$$+ (x_{v_1} - x_{v_1} x_{v_2})$$
$$+ (x_{v_1} x_{v_2} - x_{v_1} x_{v_2} x_{v_4})$$
$$+ (x_{v_1} x_{v_2} x_{v_4} - x_{v_2} x_{v_4})$$
$$+ (x_{v_2} x_{v_4} - x_{v_2} x_{v_3} x_{v_4})$$
$$+ (x_{v_2} x_{v_3} x_{v_4} - x_{v_2} x_{v_3} x_{v_4} x_{v_5})$$
$$+ (x_{v_2} x_{v_3} x_{v_4} x_{v_5} - x_{v_3} x_{v_4} x_{v_5})$$
$$+ (x_{v_3} x_{v_4} x_{v_5} - x_{v_3} x_{v_4} x_{v_5} x_{v_6})$$
$$= 1 - x_{v_3} x_{v_4} x_{v_5} x_{v_6}$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1} x_{v_2} (1 - x_{v_4}) = 0$$
$$x_{v_2} x_{v_3} (1 - x_{v_5}) = 0$$
$$x_{v_4} x_{v_5} (1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



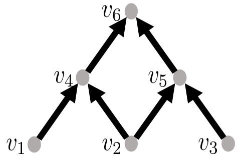
$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



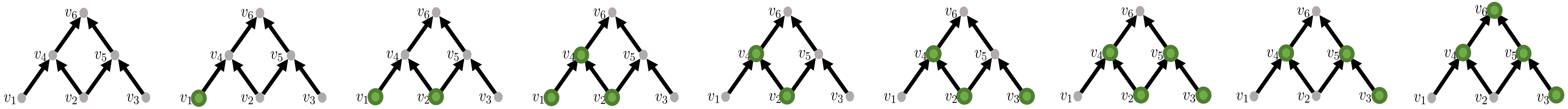
$$(1 - x_{v_1})$$
$$+ x_{v_1}(1 - x_{v_2})$$
$$+ x_{v_1} x_{v_2}(1 - x_{v_4})$$
$$+ x_{v_2} x_{v_4}(-1)(1 - x_{v_1})$$
$$+ x_{v_2} x_{v_4}(1 - x_{v_3})$$
$$+ x_{v_2} x_{v_3} x_{v_4}(1 - x_{v_5})$$
$$+ x_{v_3} x_{v_4} x_{v_5}(-1)(1 - x_{v_2})$$
$$+ x_{v_3} x_{v_4} x_{v_5}(1 - x_{v_6})$$
$$= 1 - x_{v_3} x_{v_4} x_{v_5} x_{v_6}$$

$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1} x_{v_2}(1 - x_{v_4}) = 0$$
$$x_{v_2} x_{v_3}(1 - x_{v_5}) = 0$$
$$x_{v_4} x_{v_5}(1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

Reversible pebbling to NS refutation



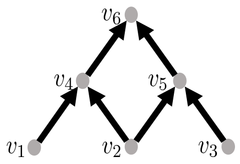
$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



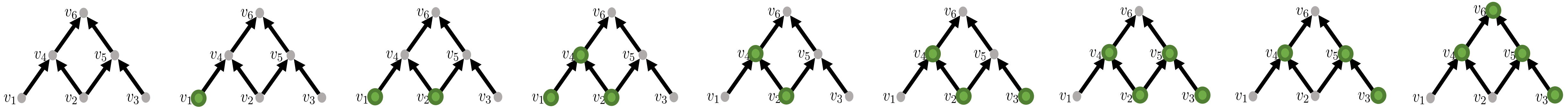
$$
\begin{aligned}
&(1 - x_{v_1}) \\
&+ x_{v_1}(1 - x_{v_2}) \\
&+ x_{v_1}x_{v_2}(1 - x_{v_4}) \\
&+ x_{v_2}x_{v_4}(-1)(1 - x_{v_1}) \\
&+ x_{v_2}x_{v_4}(1 - x_{v_3}) \\
&+ x_{v_2}x_{v_3}x_{v_4}(1 - x_{v_5}) \\
&+ x_{v_3}x_{v_4}x_{v_5}(-1)(1 - x_{v_2}) \\
&+ x_{v_3}x_{v_4}x_{v_5}(1 - x_{v_6}) \\
&+ x_{v_3}x_{v_4}x_{v_5}x_{v_6} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
1 - x_{v_1} &= 0 \\
1 - x_{v_2} &= 0 \\
1 - x_{v_3} &= 0 \\
x_{v_1}x_{v_2}(1 - x_{v_4}) &= 0 \\
x_{v_2}x_{v_3}(1 - x_{v_5}) &= 0 \\
x_{v_4}x_{v_5}(1 - x_{v_6}) &= 0 \\
x_{v_6} &= 0
\end{aligned}
$$

Reversible pebbling to NS refutation



$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



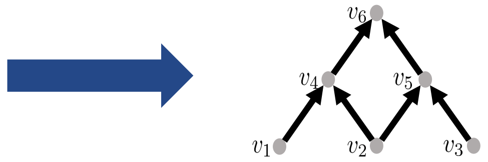
$$
\begin{aligned}
&(1 - x_{v_1}) \\
&\quad + x_{v_1}(1 - x_{v_2}) \\
&\quad\quad + x_{v_1}x_{v_2}(1 - x_{v_4}) \\
&\quad\quad\quad + x_{v_2}x_{v_4}(-1)(1 - x_{v_1}) \\
&\quad\quad\quad\quad + x_{v_2}x_{v_4}(1 - x_{v_3}) \\
&\quad\quad\quad\quad\quad + x_{v_2}x_{v_3}x_{v_4}(1 - x_{v_5}) \\
&\quad\quad\quad\quad\quad\quad + x_{v_3}x_{v_4}x_{v_5}(-1)(1 - x_{v_2}) \\
&\quad\quad\quad\quad\quad\quad\quad + x_{v_3}x_{v_4}x_{v_5}(1 - x_{v_6}) \\
&\quad\quad\quad\quad\quad\quad\quad\quad + x_{v_3}x_{v_4}x_{v_5}x_{v_6} \\
&\quad\quad\quad\quad\quad\quad\quad\quad = 1
\end{aligned}
$$

$$
\begin{aligned}
1 - x_{v_1} &= 0 \\
1 - x_{v_2} &= 0 \\
1 - x_{v_3} &= 0 \\
x_{v_1}x_{v_2}(1 - x_{v_4}) &= 0 \\
x_{v_2}x_{v_3}(1 - x_{v_5}) &= 0 \\
x_{v_4}x_{v_5}(1 - x_{v_6}) &= 0 \\
x_{v_6} &= 0
\end{aligned}
$$

NS size $= 2 \cdot \frac{\text{pebbling time}}{2} + 1 = 2 \cdot 8 + 1$
NS degree $=$ pebbling space $= 4$

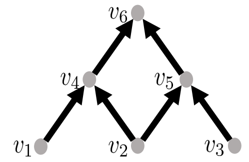
NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$

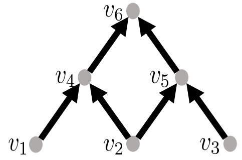


$$1 - x_{v_1} = 0$$
$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
$$x_{v_1} x_{v_2} (1 - x_{v_4}) = 0$$
$$x_{v_2} x_{v_3} (1 - x_{v_5}) = 0$$
$$x_{v_4} x_{v_5} (1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

$$(1 - x_{v_2} x_{v_4})(1 - x_{v_1}) + (x_{v_1} - x_{v_3} x_{v_4} x_{v_5})(1 - x_{v_2}) + x_{v_2} x_{v_4}(1 - x_{v_3}) + x_{v_1} x_{v_2}(1 - x_{v_4}) + x_{v_4} x_{v_2} x_{v_3}(1 - x_{v_5}) + x_{v_3} x_{v_4} x_{v_5}(1 - x_{v_6}) + x_{v_3} x_{v_4} x_{v_5} x_{v_6} = 1$$

NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



For simplicity, assume $\mathbb{F}_2$

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$$1 - x_{v_2} = 0$$
$$1 - x_{v_3} = 0$$
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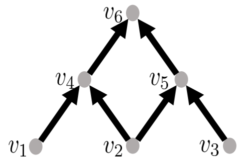
$$x_{v_2} x_{v_3}(1 - x_{v_5}) = 0$$
$$x_{v_4} x_{v_5}(1 - x_{v_6}) = 0$$
$$x_{v_6} = 0$$

$$(1 - x_{v_2} x_{v_4})(1 - x_{v_1}) + (x_{v_1} - x_{v_3} x_{v_4} x_{v_5})(1 - x_{v_2}) + x_{v_2} x_{v_4}(1 - x_{v_3}) + x_{v_1} x_{v_2}(1 - x_{v_4}) + x_{v_4} x_{v_2} x_{v_3}(1 - x_{v_5}) + x_{v_3} x_{v_4} x_{v_5}(1 - x_{v_6}) + x_{v_3} x_{v_4} x_{v_5} x_{v_6} = 1$$

NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



For simplicity, assume $\mathbb{F}_2$

$$1 + x_{v_1} = 0$$
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$$1 + x_{v_3} = 0$$
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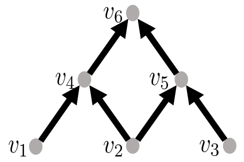
$$+ x_{v_4} x_{v_2} x_{v_3} (1 + x_{v_5}) + x_{v_3} x_{v_4} x_{v_5} (1 + x_{v_6}) + x_{v_3} x_{v_4} x_{v_5} x_{v_6} = 1$$

NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



∀ monomials x_W in proof, add node W

For simplicity, assume $\mathbb{F}_2$

$$1 + x_{v_1} = 0$$
$$1 + x_{v_2} = 0$$
$$1 + x_{v_3} = 0$$
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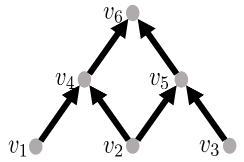
$$x_{v_2} x_{v_3}(1 + x_{v_5}) = 0$$
$$x_{v_4} x_{v_5}(1 + x_{v_6}) = 0$$
$$x_{v_6} = 0$$

$$(1 + x_{v_2} x_{v_4})(1 + x_{v_1}) + (x_{v_1} + x_{v_3} x_{v_4} x_{v_5})(1 + x_{v_2}) + x_{v_2} x_{v_4}(1 + x_{v_3}) + x_{v_1} x_{v_2}(1 + x_{v_4})$$
$$+ x_{v_4} x_{v_2} x_{v_3}(1 + x_{v_5}) + x_{v_3} x_{v_4} x_{v_5}(1 + x_{v_6}) + x_{v_3} x_{v_4} x_{v_5} x_{v_6} = 1$$

NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



$\forall$ monomials x_W in proof, add node W

$$x_W = \prod_{v \in W} x_v$$

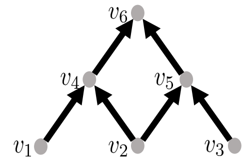
For simplicity, assume $\mathbb{F}_2$

$$1 + x_{v_1} = 0$$
$$1 + x_{v_2} = 0$$
$$1 + x_{v_3} = 0$$
$$x_{v_1} x_{v_2} (1 + x_{v_4}) = 0$$
$$x_{v_2} x_{v_3} (1 + x_{v_5}) = 0$$
$$x_{v_4} x_{v_5} (1 + x_{v_6}) = 0$$
$$x_{v_6} = 0$$

$$(1 + x_{v_2} x_{v_4})(1 + x_{v_1}) + (x_{v_1} + x_{v_3} x_{v_4} x_{v_5})(1 + x_{v_2}) + x_{v_2} x_{v_4} (1 + x_{v_3}) + x_{v_1} x_{v_2} (1 + x_{v_4}) + x_{v_4} x_{v_2} x_{v_3} (1 + x_{v_5}) + x_{v_3} x_{v_4} x_{v_5} (1 + x_{v_6}) + x_{v_3} x_{v_4} x_{v_5} x_{v_6} = 1$$

NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



$\forall$ monomials x_W in proof, add node W

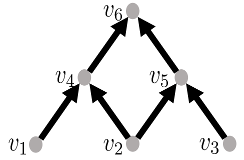
For simplicity, assume $\mathbb{F}_2$

$$\begin{aligned} 1 + x_{v_1} &= 0 \\ 1 + x_{v_2} &= 0 \\ 1 + x_{v_3} &= 0 \\ x_{v_1}x_{v_2}(1 + x_{v_4}) &= 0 \\ x_{v_2}x_{v_3}(1 + x_{v_5}) &= 0 \\ x_{v_4}x_{v_5}(1 + x_{v_6}) &= 0 \\ x_{v_6} &= 0 \end{aligned}$$

$$(1 + x_{v_2}x_{v_4})(1 + x_{v_1}) + (x_{v_1} + x_{v_3}x_{v_4}x_{v_5})(1 + x_{v_2}) + x_{v_2}x_{v_4}(1 + x_{v_3}) + x_{v_1}x_{v_2}(1 + x_{v_4})$$
$$+ x_{v_4}x_{v_2}x_{v_3}(1 + x_{v_5}) + x_{v_3}x_{v_4}x_{v_5}(1 + x_{v_6}) + x_{v_3}x_{v_4}x_{v_5}x_{v_6} = 1$$

NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



$\forall$ monomials x_W in proof, add node W

For simplicity, assume $\mathbb{F}_2$

$\emptyset$ | $\{v_1\}$ | $\{v_1, v_2\}$ | $\{v_2, v_4\}$ | $\{v_1, v_2, v_4\}$ | $\{v_2, v_3, v_4\}$ | $\{v_3, v_4, v_5\}$ | $\{v_2, v_3, v_4, v_5\}$ | $\{v_3, v_4, v_5, v_6\}$

$$1 + x_{v_1} = 0$$
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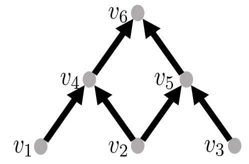
$$(1 + x_{v_2}x_{v_4})(1 + x_{v_1}) + (x_{v_1} + x_{v_3}x_{v_4}x_{v_5})(1 + x_{v_2}) + x_{v_2}x_{v_4}(1 + x_{v_3}) + x_{v_1}x_{v_2}(1 + x_{v_4})$$
$$+ x_{v_4}x_{v_2}x_{v_3}(1 + x_{v_5}) + x_{v_3}x_{v_4}x_{v_5}(1 + x_{v_6}) + x_{v_3}x_{v_4}x_{v_5}x_{v_6} = 1$$

NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1 \implies$$



$\forall$ monomials x_W in proof, add node W

$\forall v, \forall$ monomials $x_W \in Q_v$, add edge $(W \cup \text{pred}(v), W \cup \text{pred}(v) \cup \{v\})$

For simplicity, assume $\mathbb{F}_2$

$\emptyset$

$\{v_1\}$

$\{v_1, v_2\}$

$\{v_2, v_4\}$

$\{v_1, v_2, v_4\}$

$\{v_2, v_3, v_4\}$

$\{v_3, v_4, v_5\}$

$\{v_2, v_3, v_4, v_5\}$

$\{v_3, v_4, v_5, v_6\}$

$$1 + x_{v_1} = 0$$
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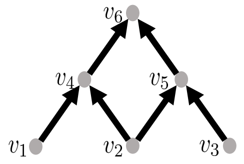
$$(1 + x_{v_2} x_{v_4})(1 + x_{v_1}) + (x_{v_1} + x_{v_3} x_{v_4} x_{v_5})(1 + x_{v_2}) + x_{v_2} x_{v_4}(1 + x_{v_3}) + x_{v_1} x_{v_2}(1 + x_{v_4})$$
$$+ x_{v_4} x_{v_2} x_{v_3}(1 + x_{v_5}) + x_{v_3} x_{v_4} x_{v_5}(1 + x_{v_6}) + x_{v_3} x_{v_4} x_{v_5} x_{v_6} = 1$$

NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



$\forall$ monomials x_W in proof, add node W

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For simplicity, assume $\mathbb{F}_2$

$\emptyset$ $\{v_1\}$ $\{v_1, v_2\}$ $\{v_2, v_4\}$ $\{v_1, v_2, v_4\}$ $\{v_2, v_3, v_4\}$ $\{v_3, v_4, v_5\}$ $\{v_2, v_3, v_4, v_5\}$ $\{v_3, v_4, v_5, v_6\}$

$$1 + x_{v_1} = 0$$
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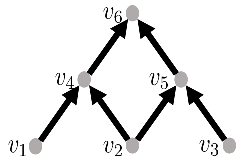
NS refutation to reversible pebbling

$$\sum_{v} Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



$\forall$ monomials x_W in proof, add node W

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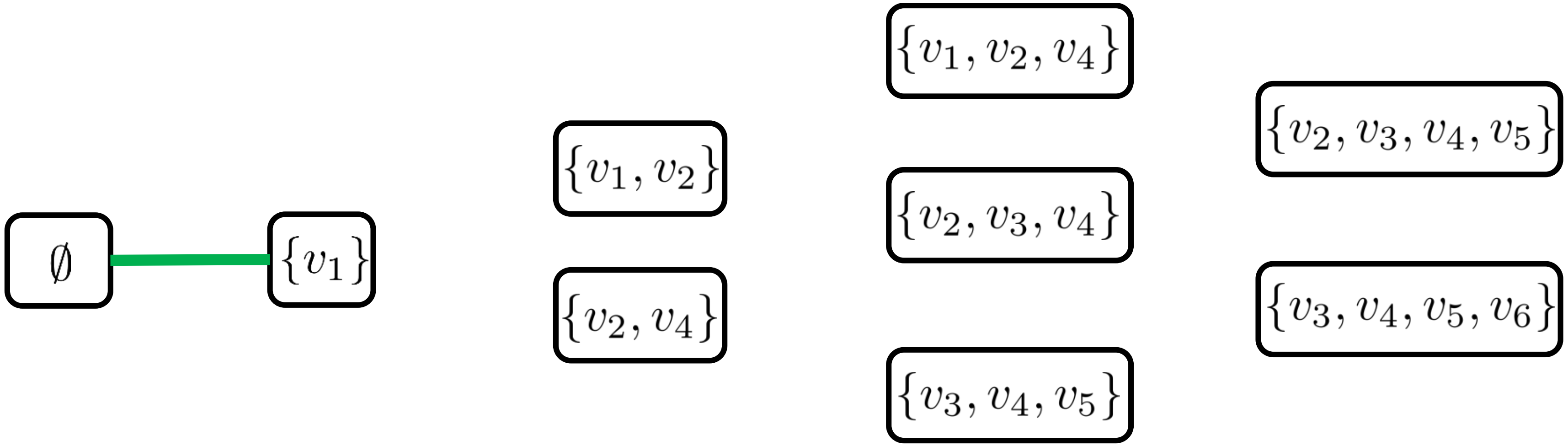
For simplicity, assume $\mathbb{F}_2$



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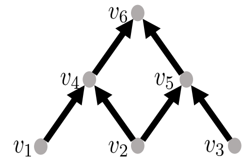
NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



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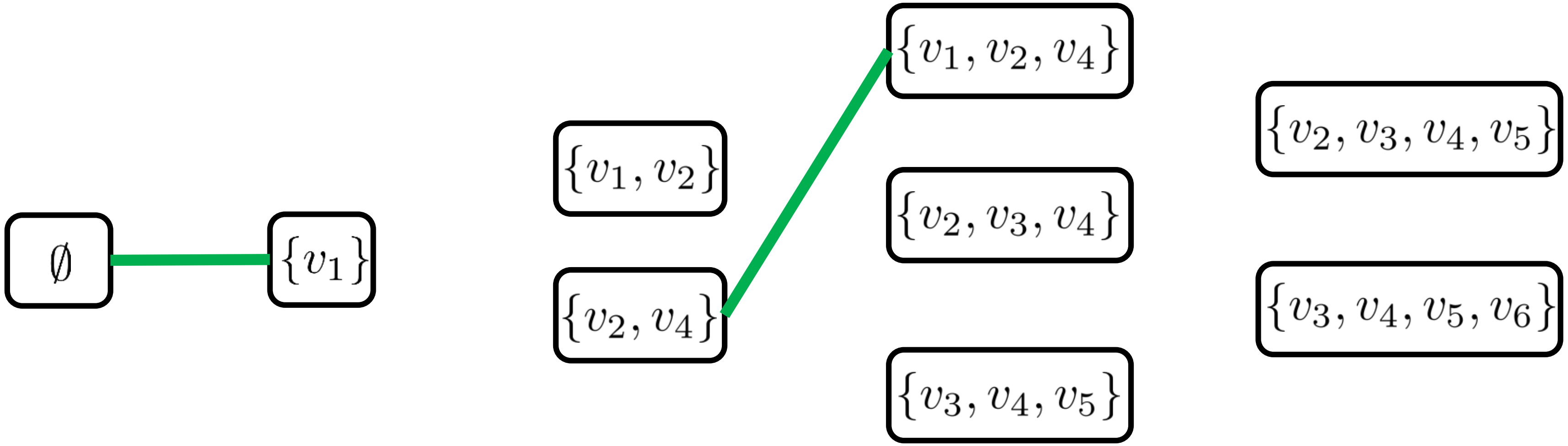

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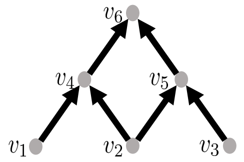
NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\text{sink}} A_{\text{sink}} = 1$$



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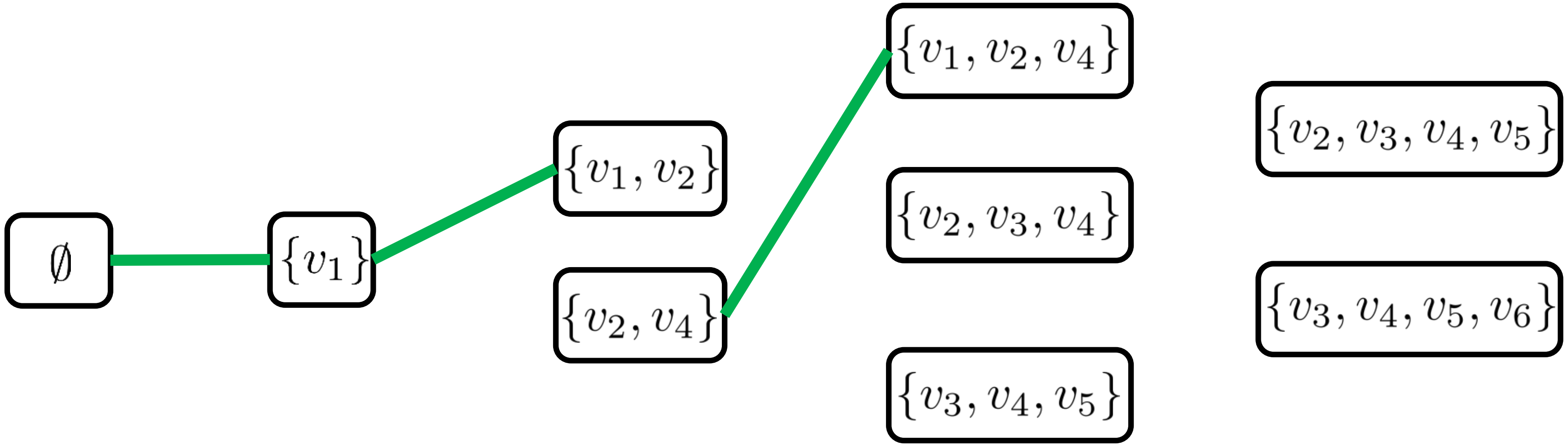
For simplicity, assume $\mathbb{F}_2$



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$$x_{v_6} = 0$$

$$(1 + x_{v_2} x_{v_4})(1 + x_{v_1}) + (x_{v_1} + x_{v_3} x_{v_4} x_{v_5})(1 + x_{v_2}) + x_{v_2} x_{v_4}(1 + x_{v_3}) + x_{v_1} x_{v_2}(1 + x_{v_4})$$
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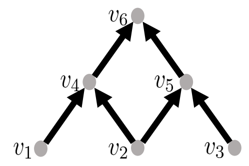
NS refutation to reversible pebbling

$$\sum_v Q_v A_v + Q_{\mathsf{sink}} A_{\mathsf{sink}} = 1$$



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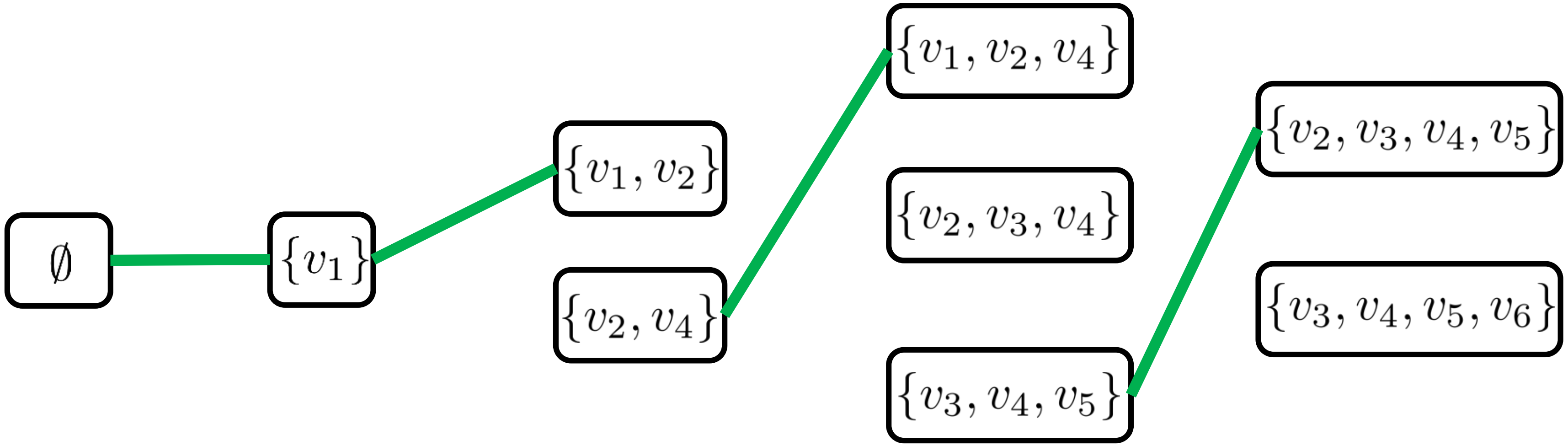
For simplicity, assume $\mathbb{F}_2$



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NS refutation to reversible pebbling

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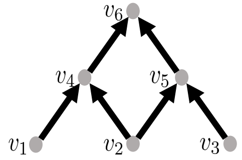

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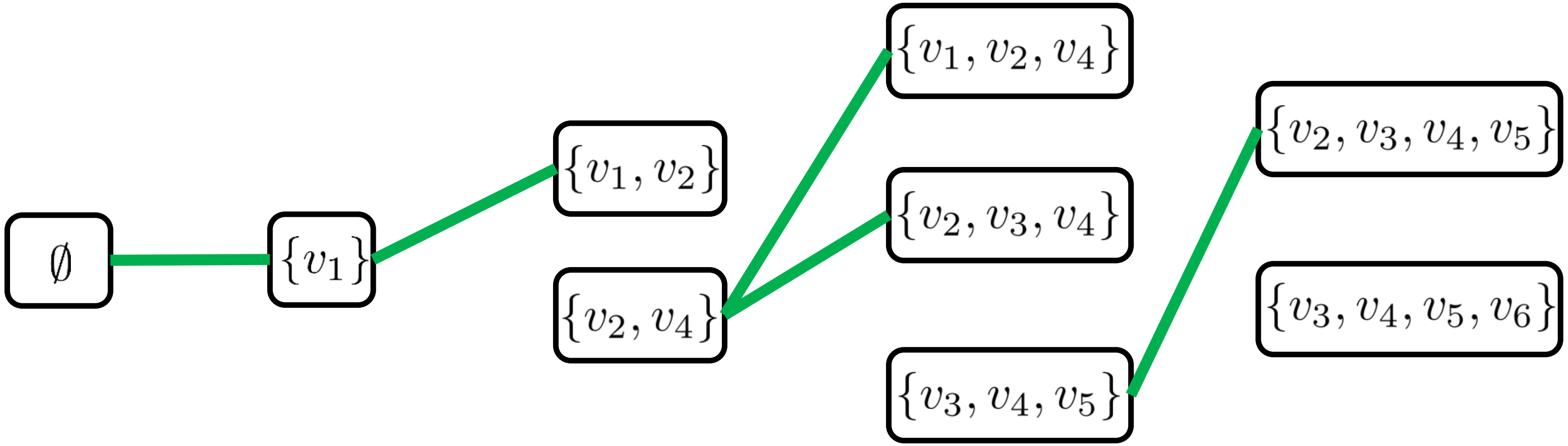

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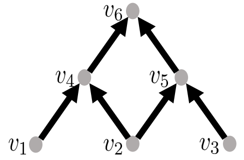

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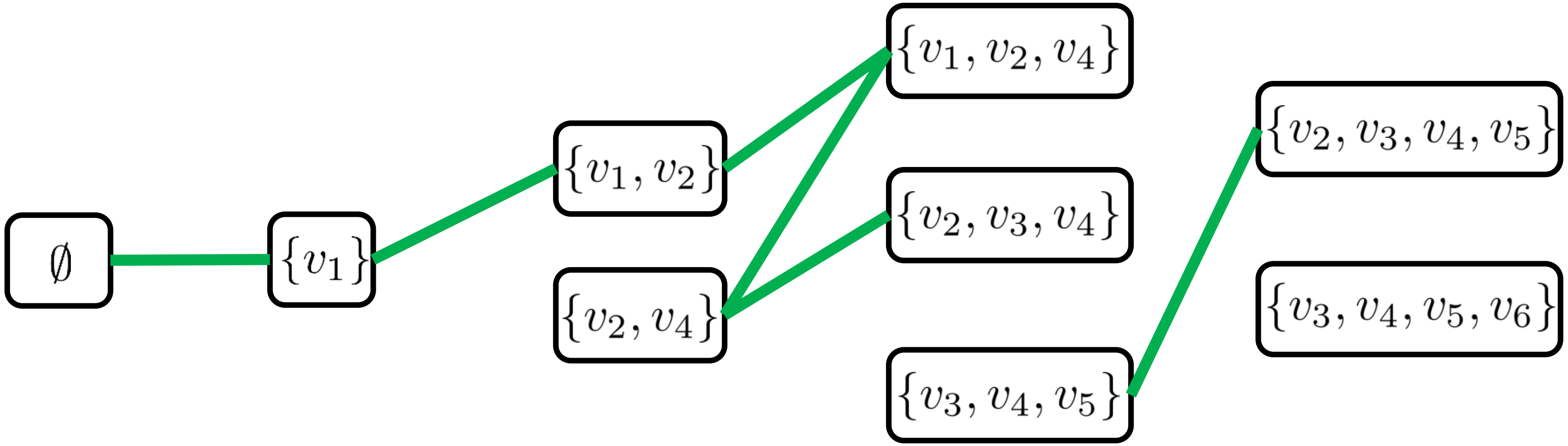

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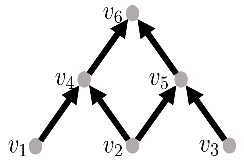

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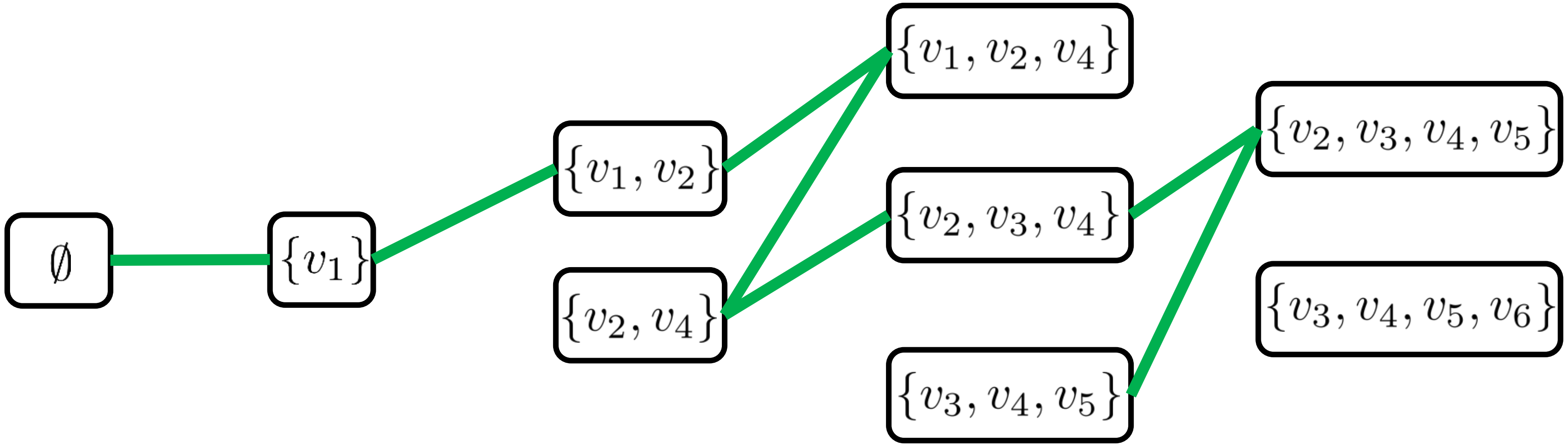

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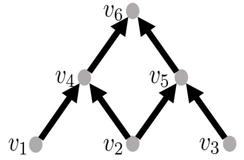

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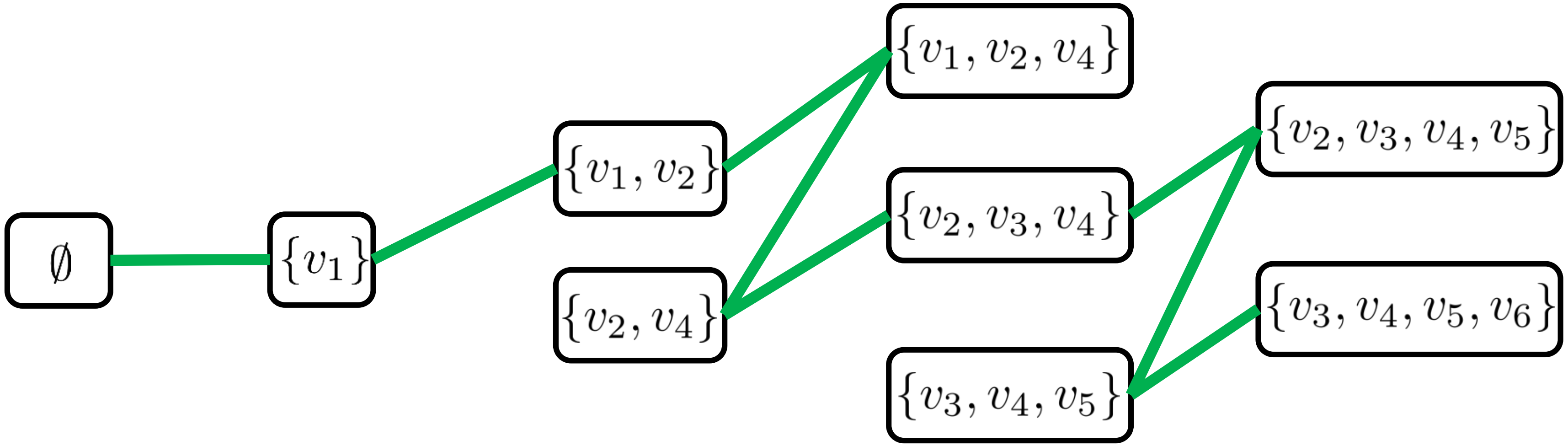

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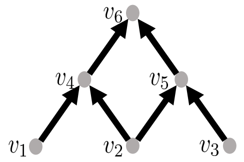

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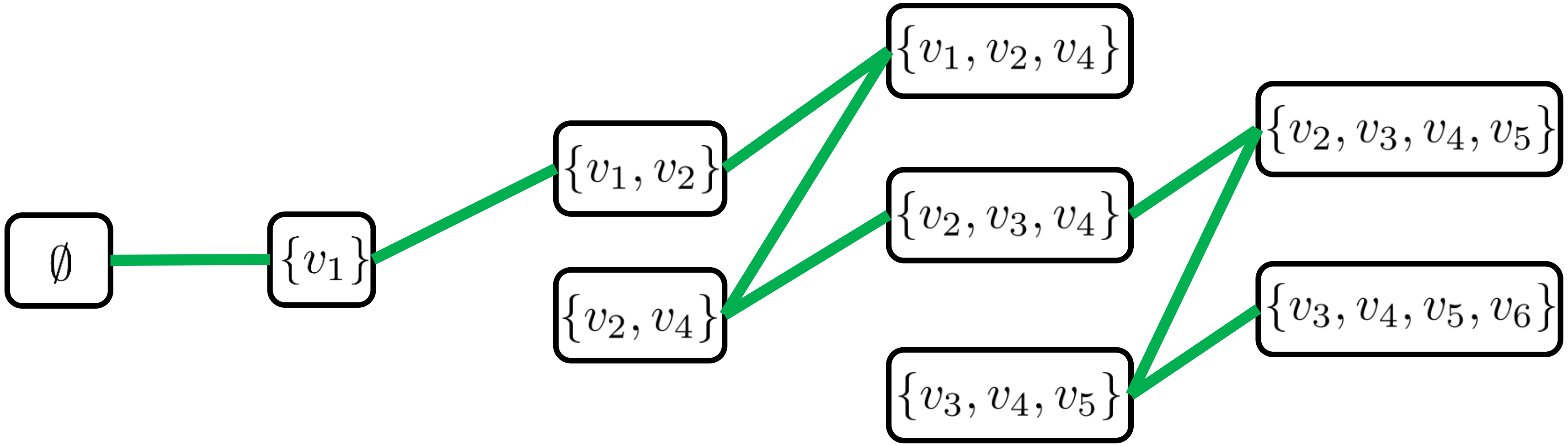

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NS refutation to reversible pebbling

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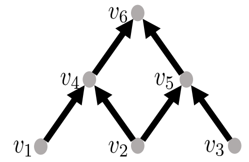

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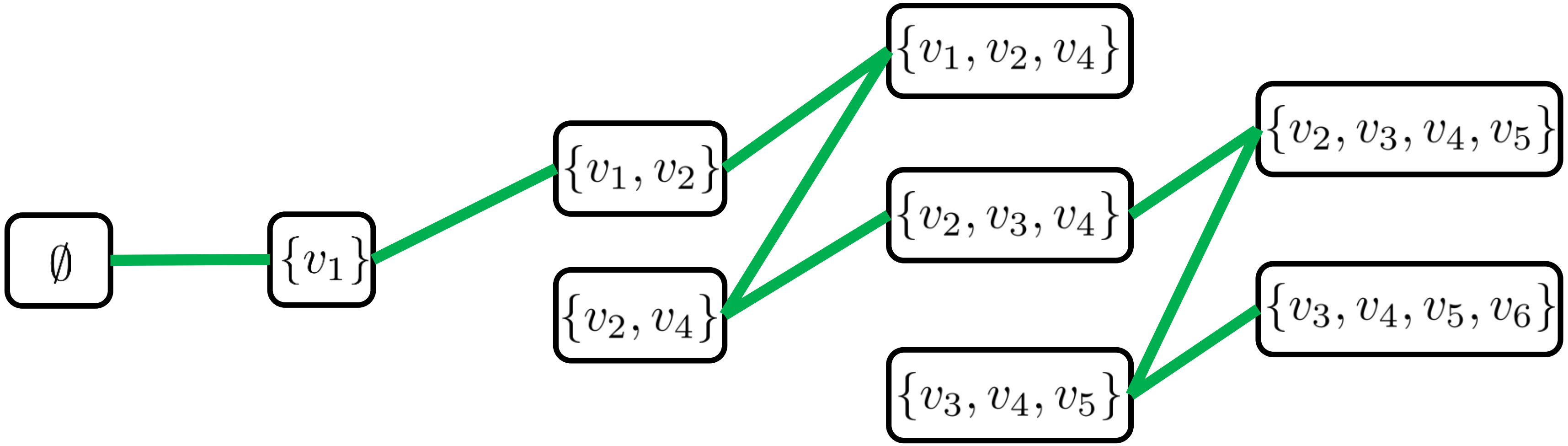

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Edges correspond to valid pebbling moves

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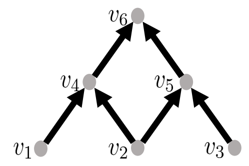

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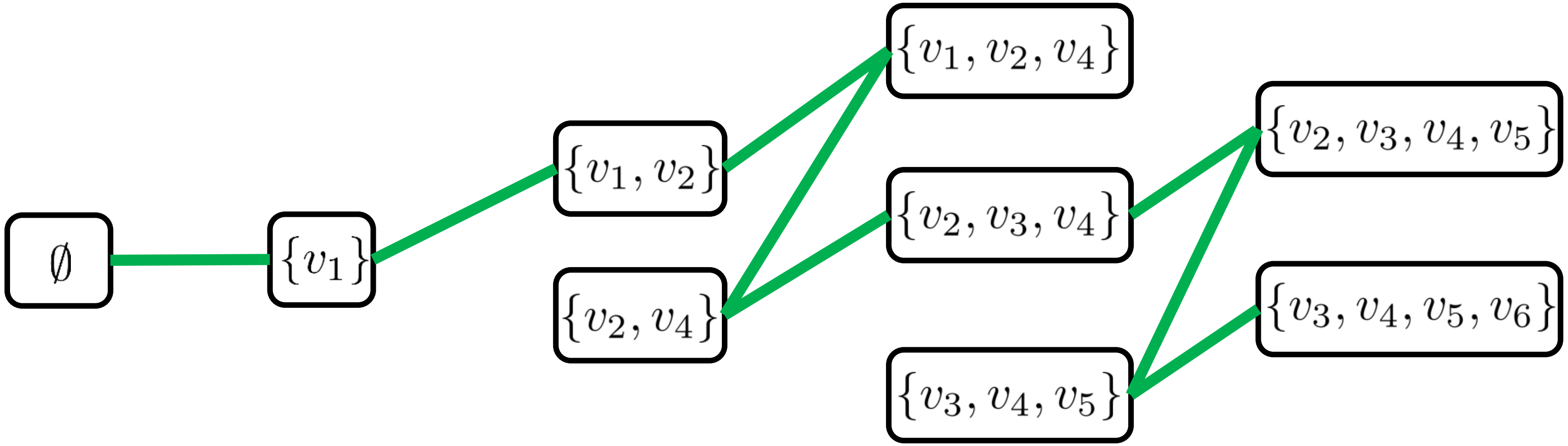

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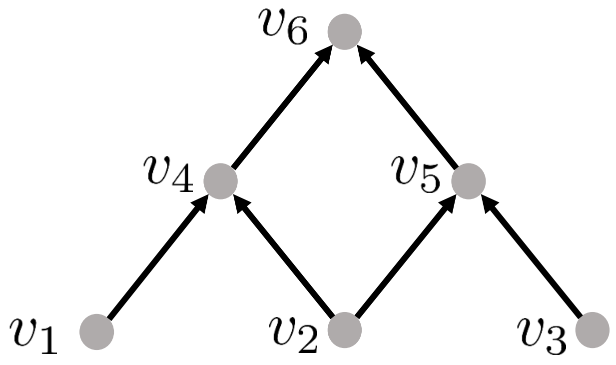

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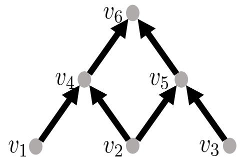

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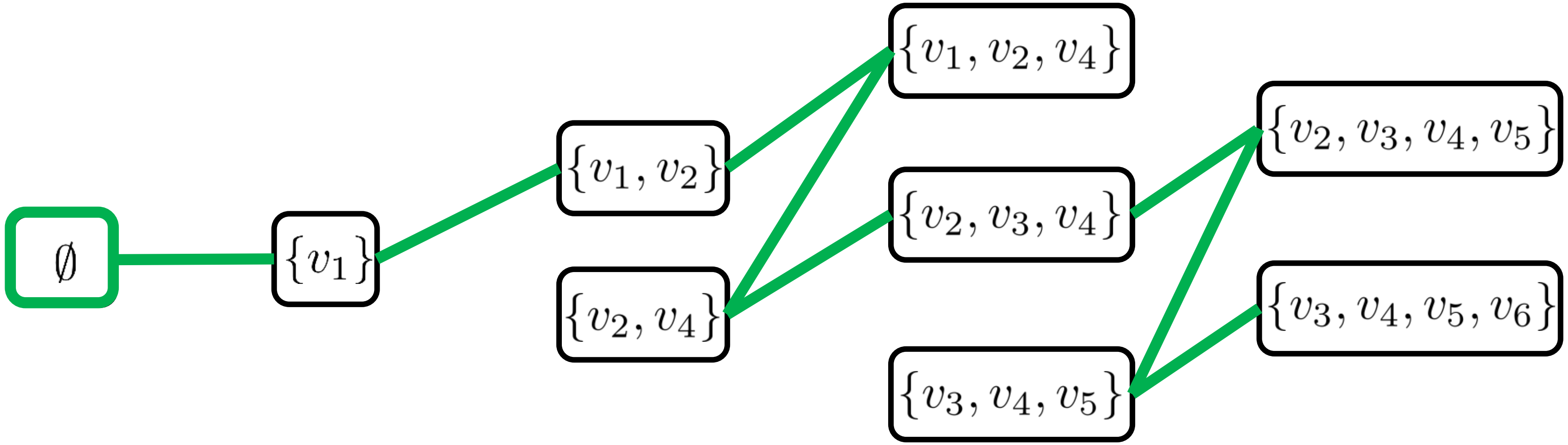

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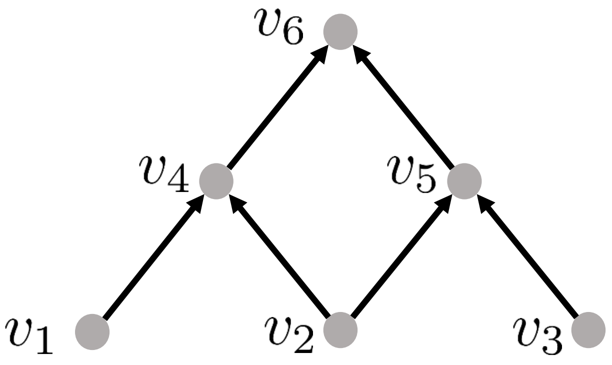

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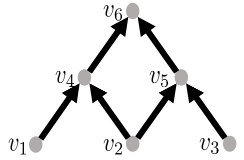

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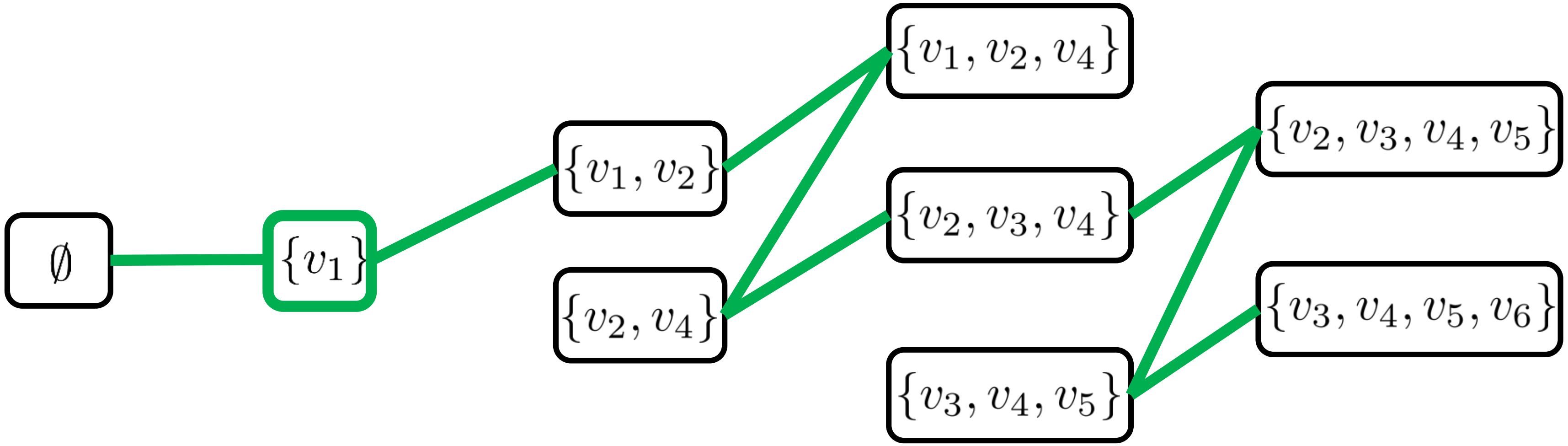

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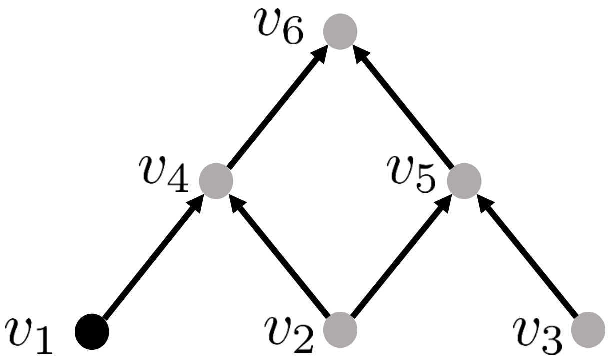

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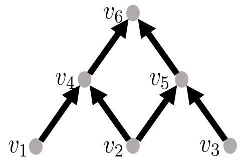

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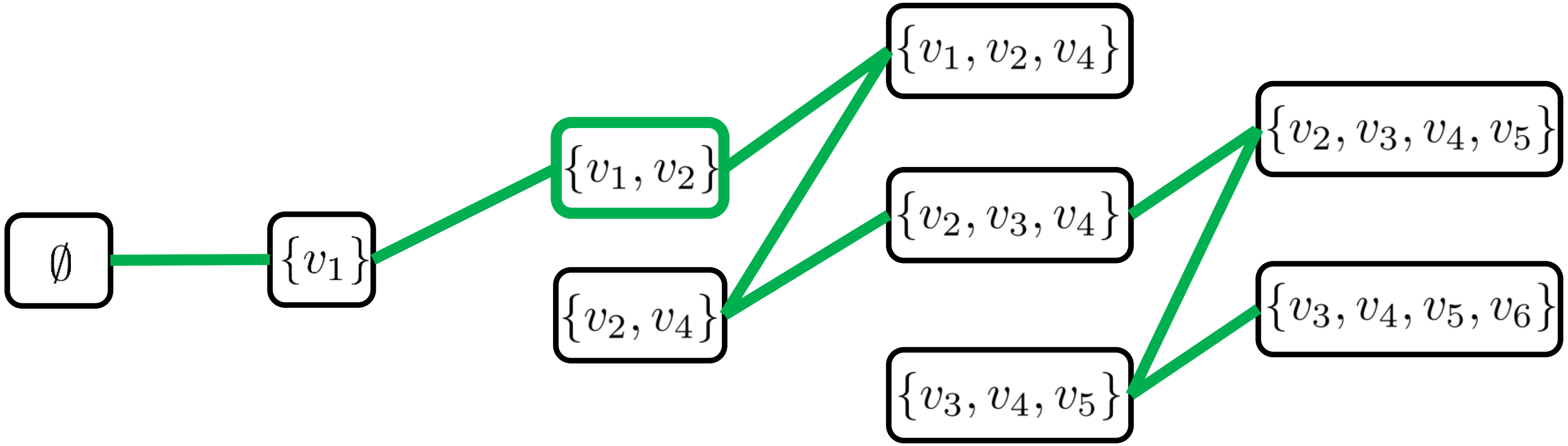

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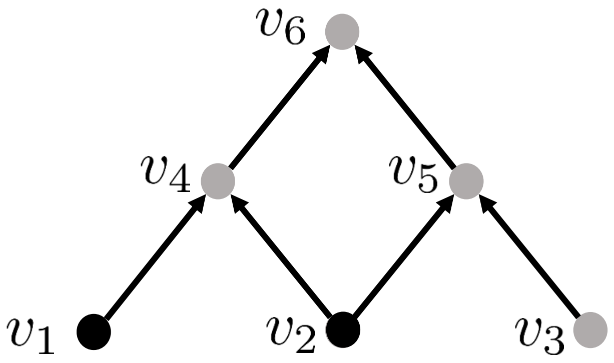

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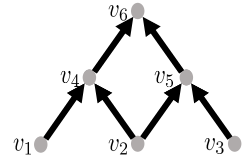

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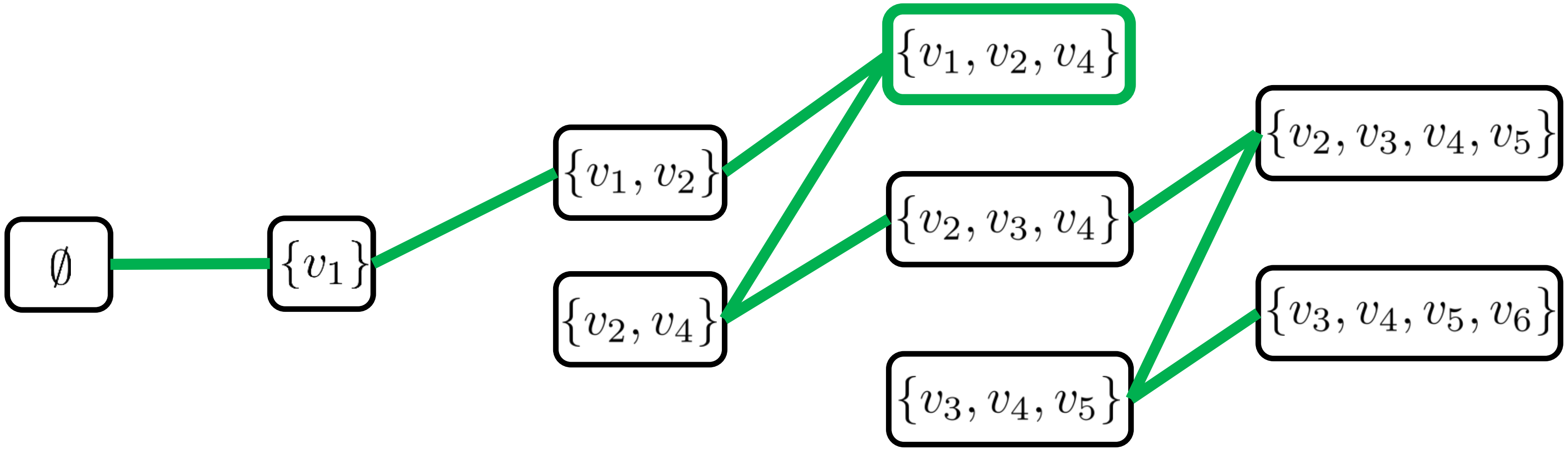

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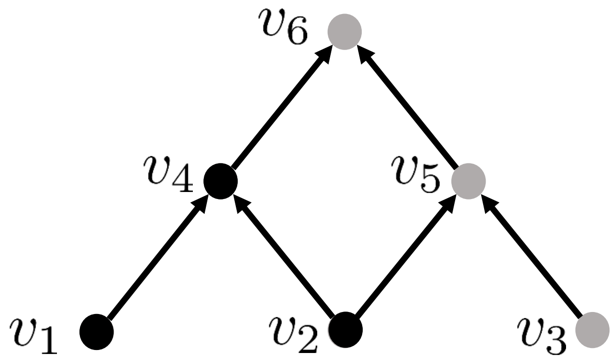

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NS refutation to reversible pebbling

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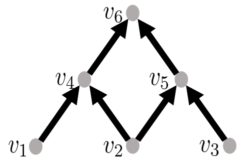

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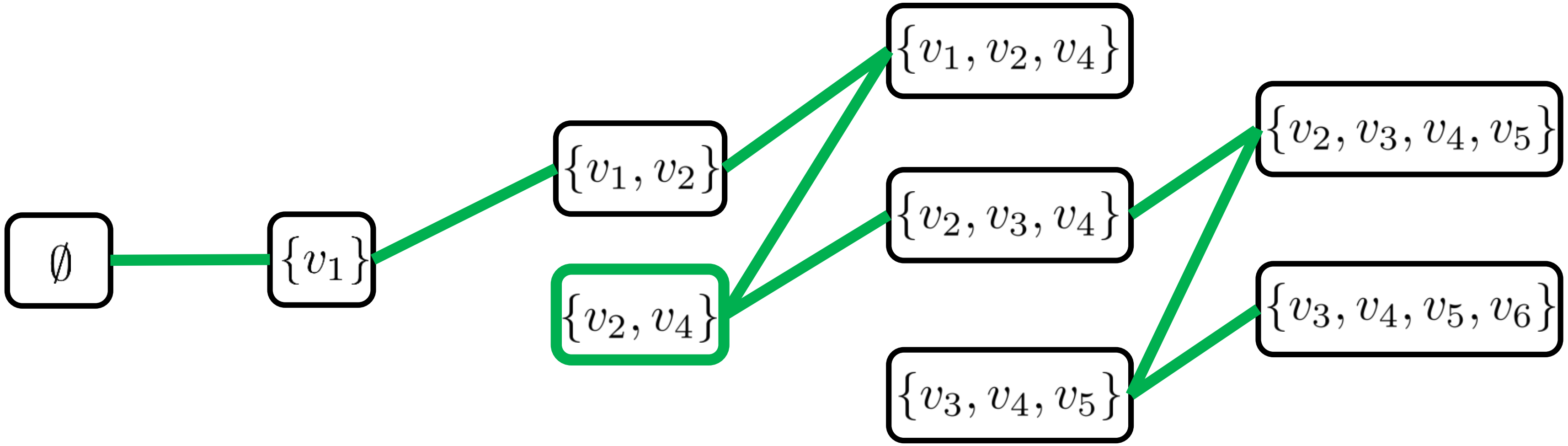

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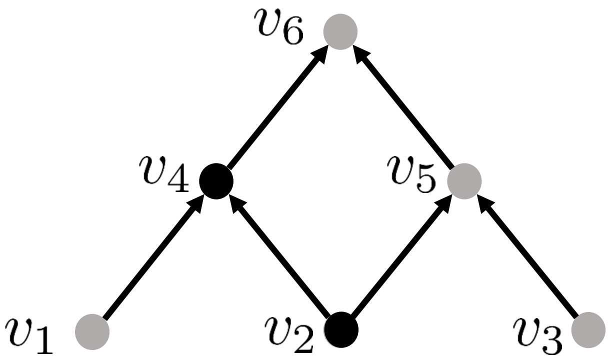

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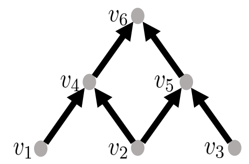

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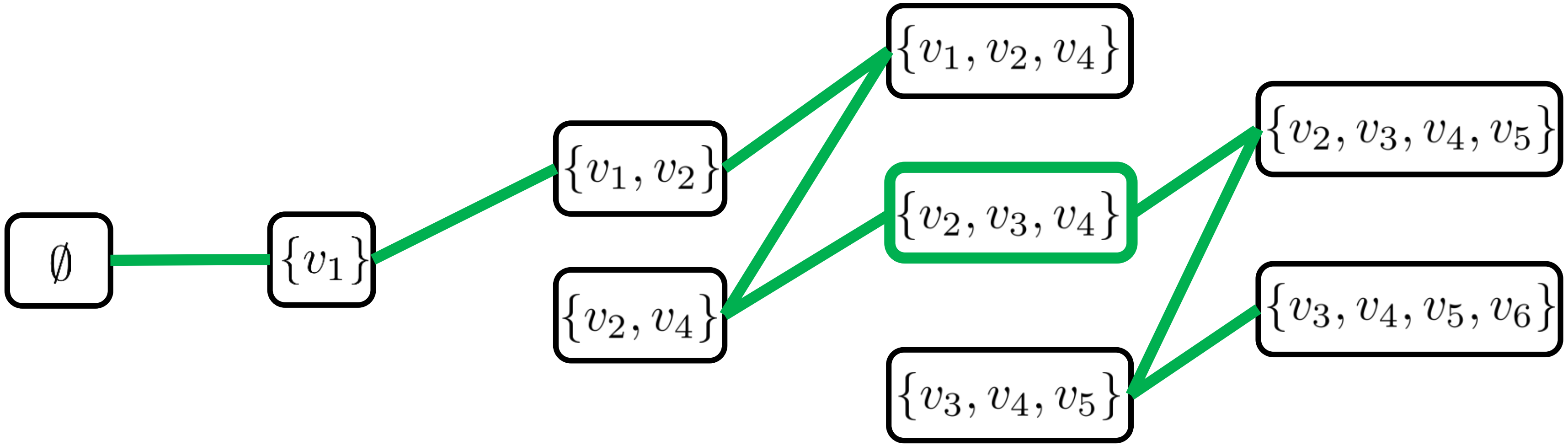

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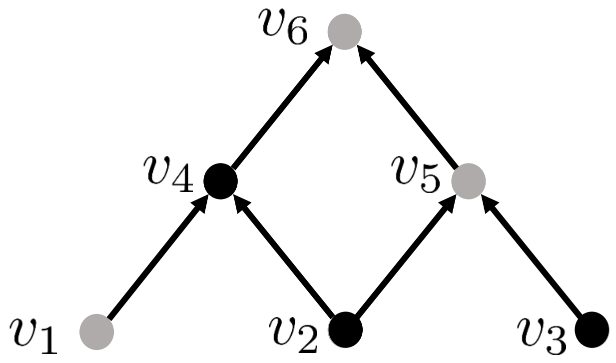

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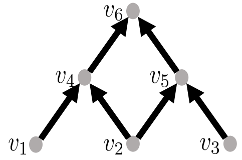

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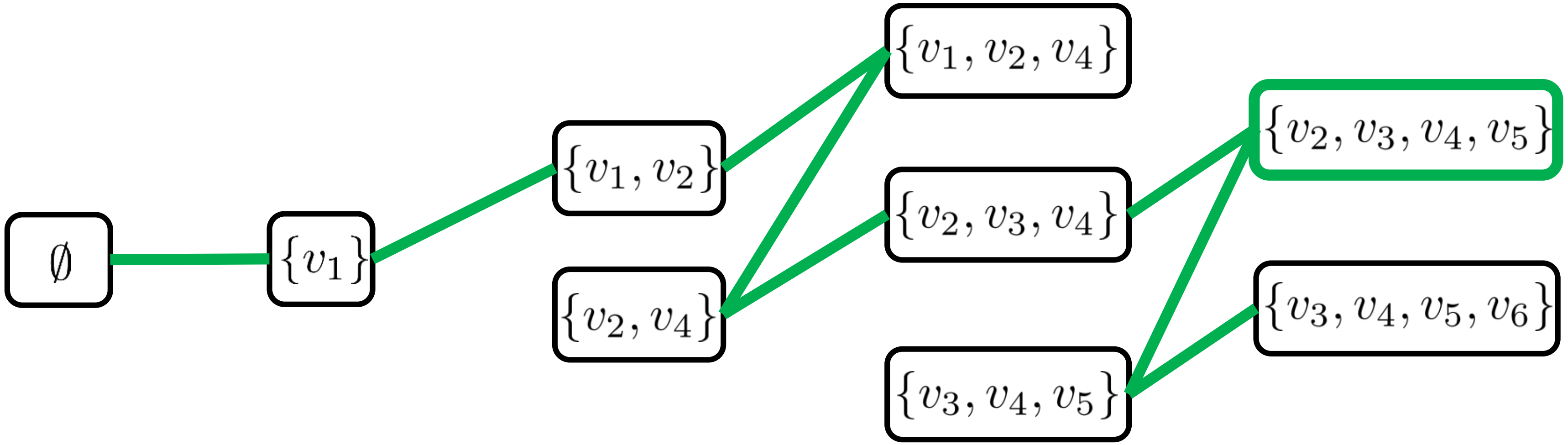

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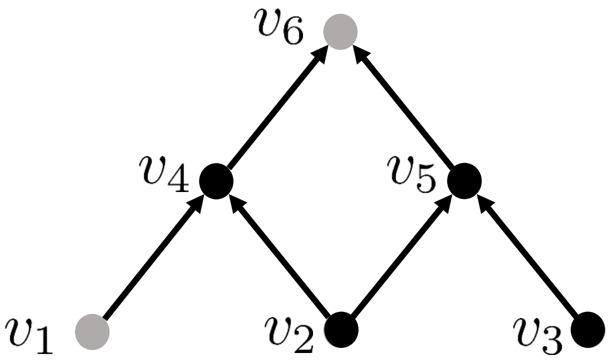

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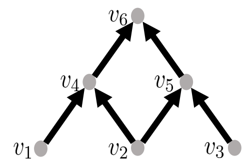

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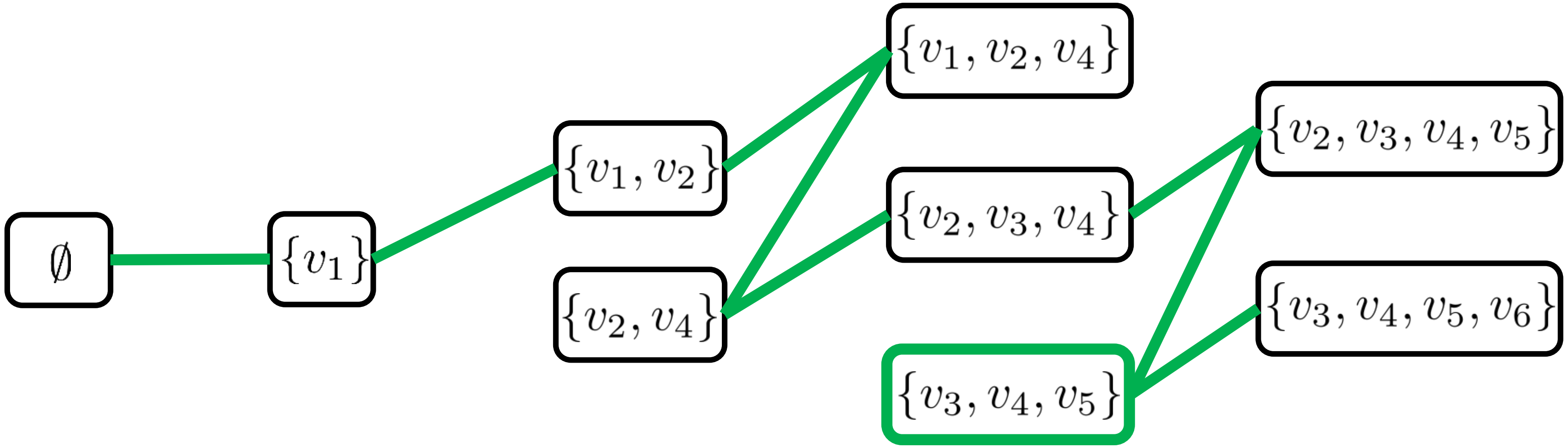

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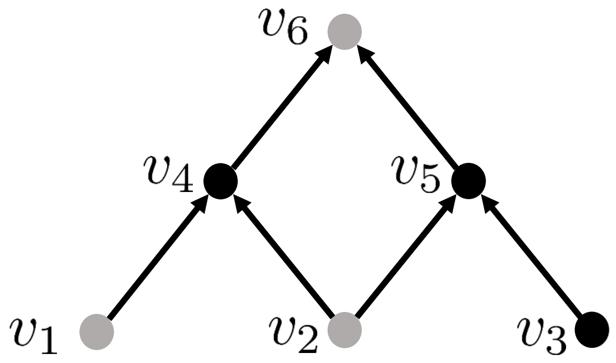

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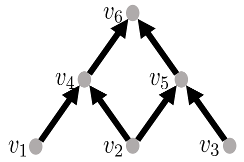

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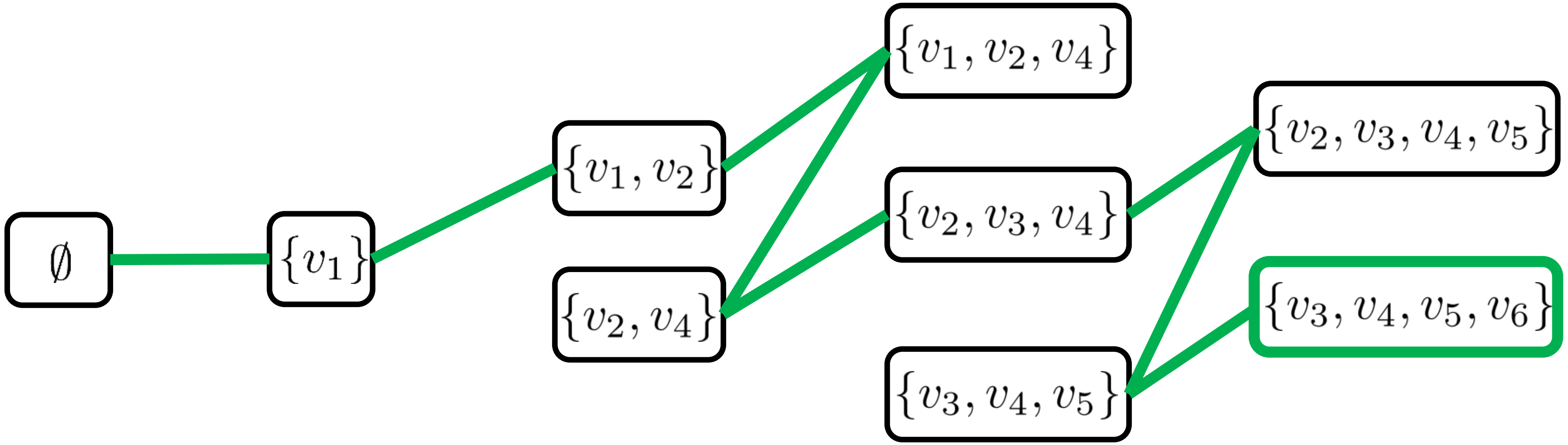

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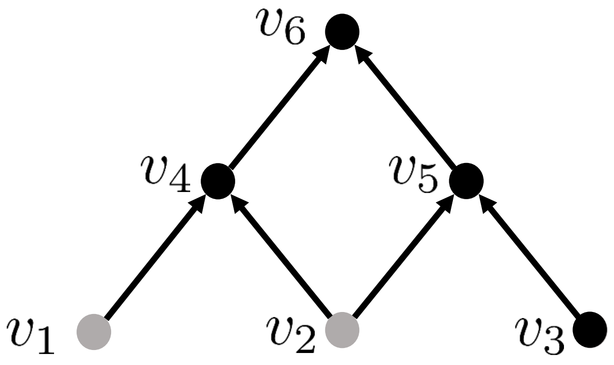

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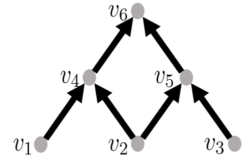

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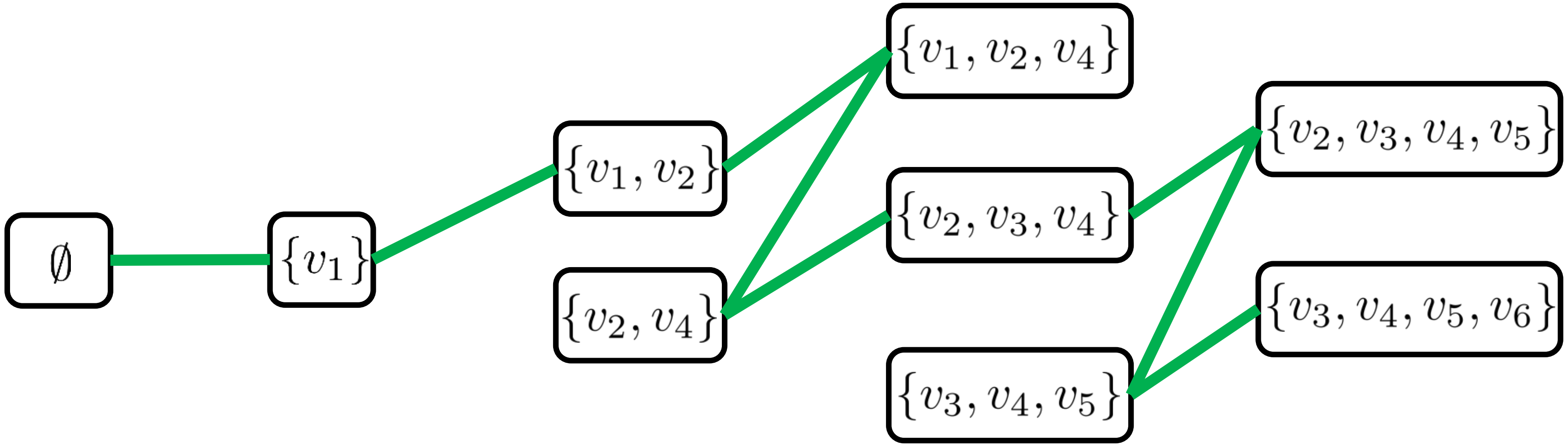

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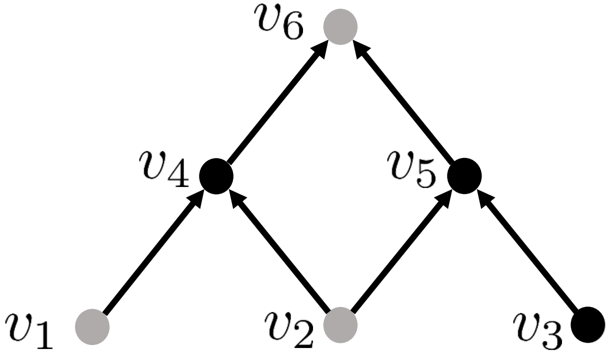

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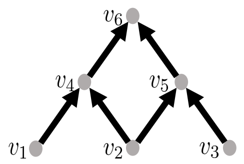

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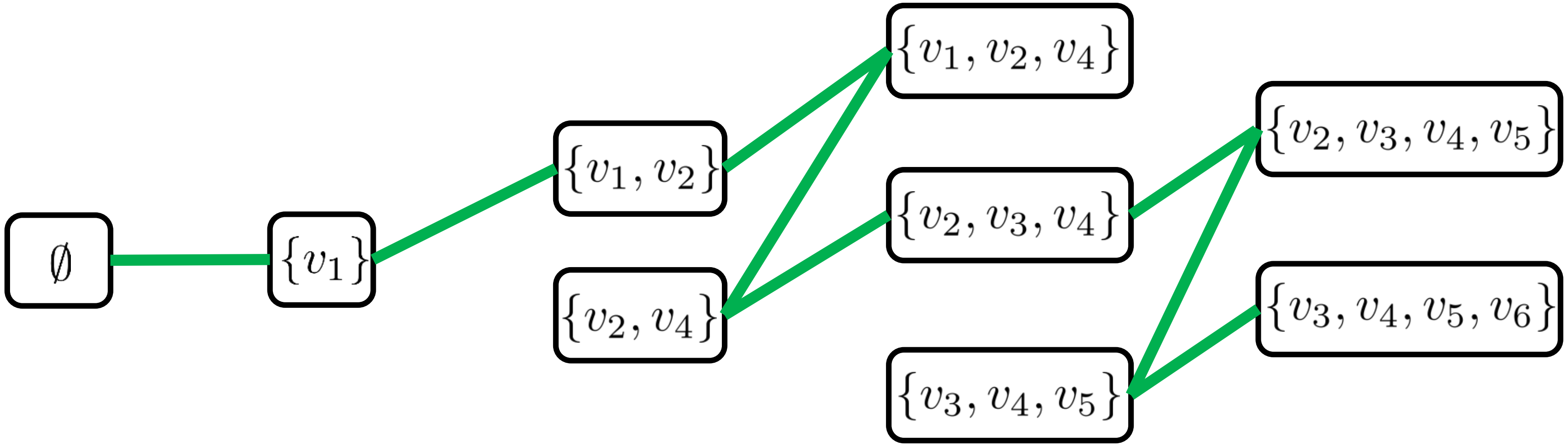

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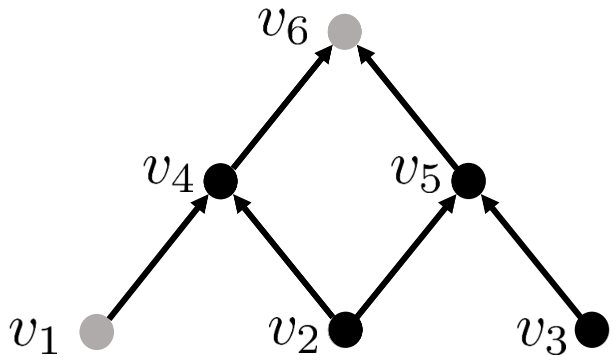

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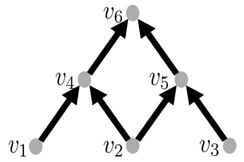

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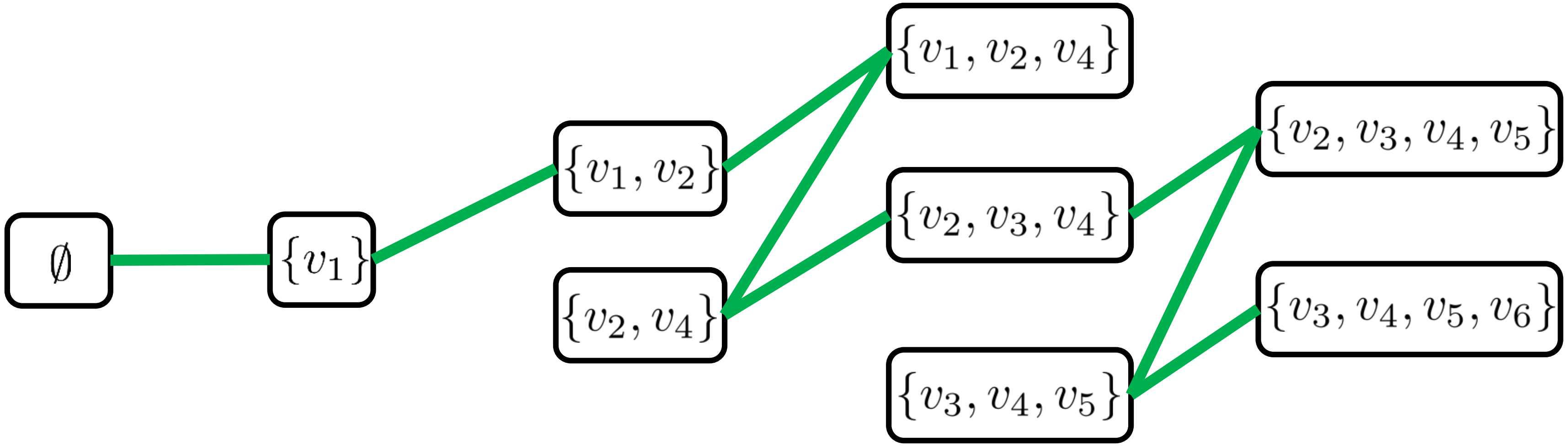

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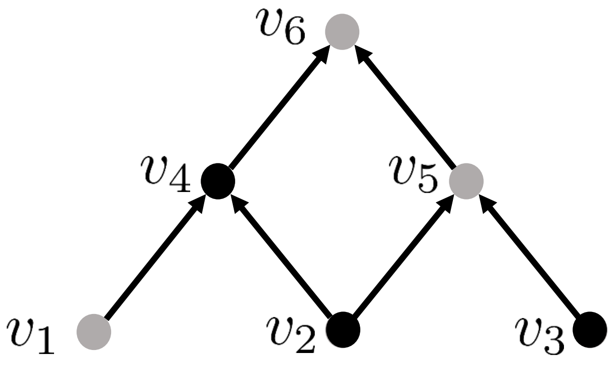

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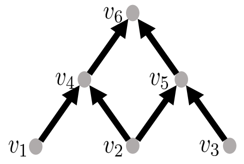

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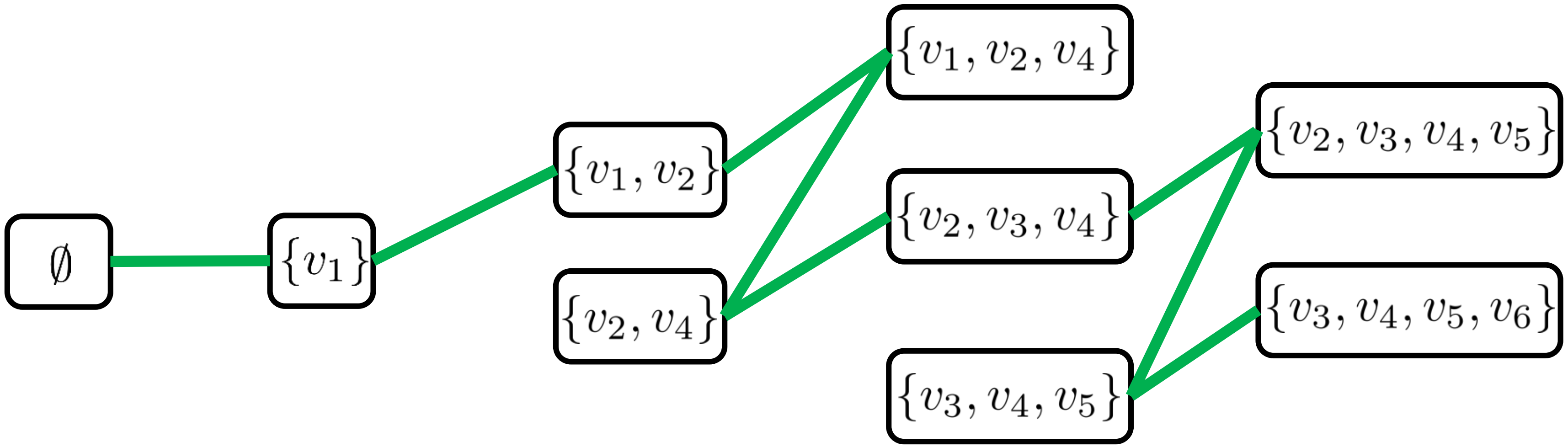

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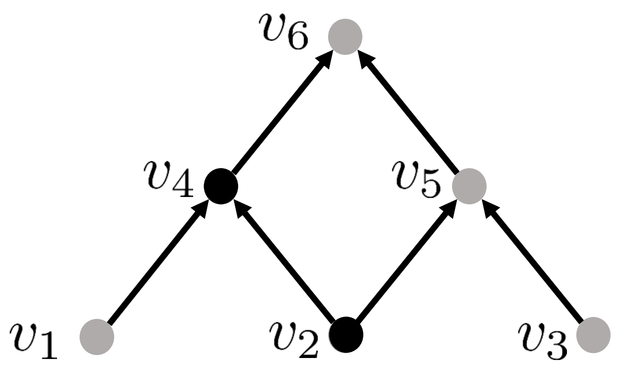

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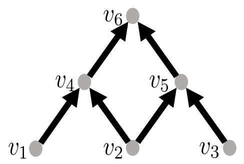

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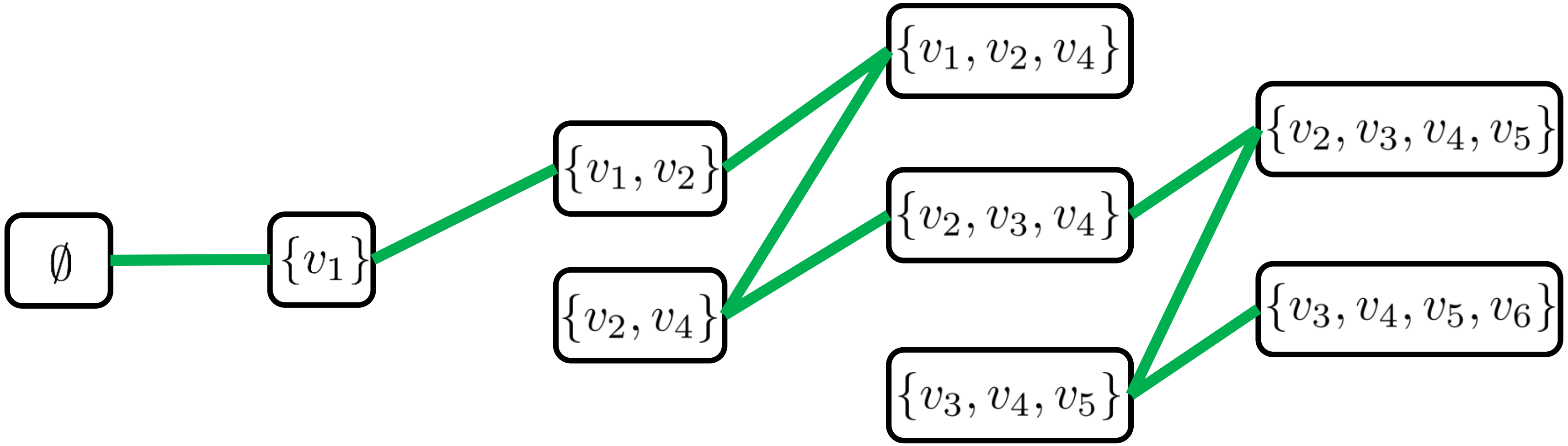

$$
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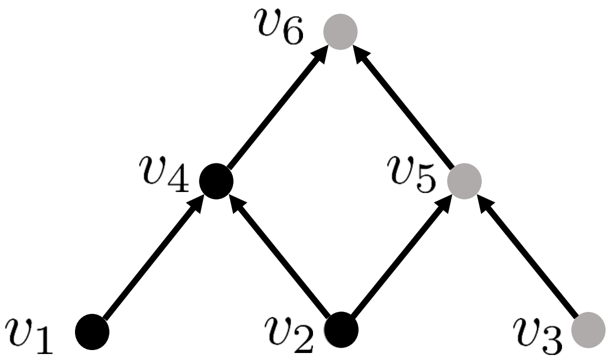

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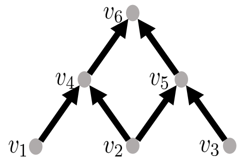

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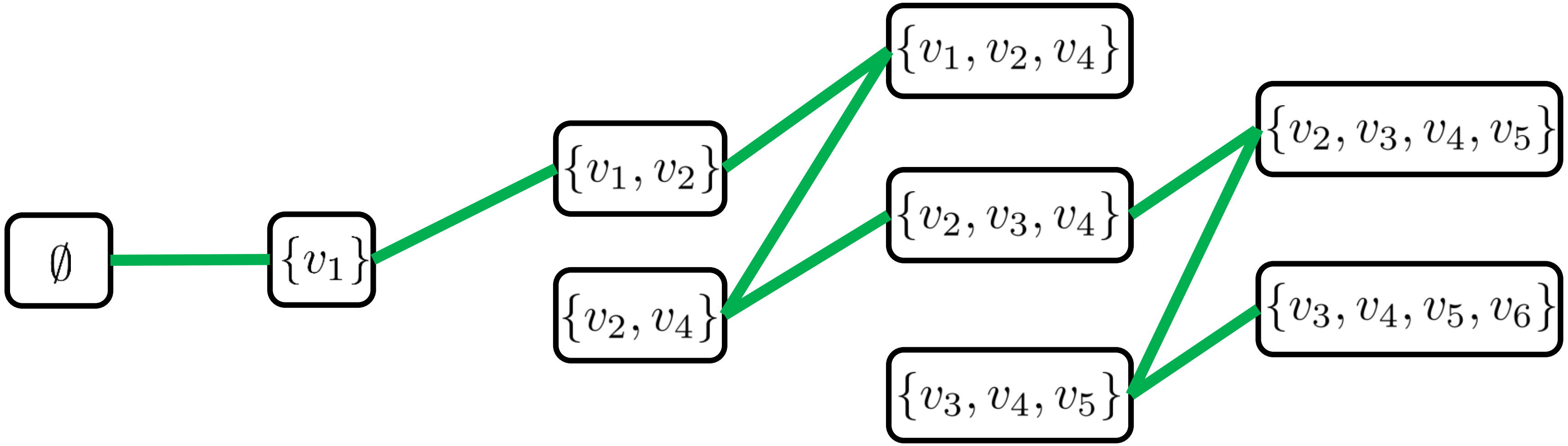

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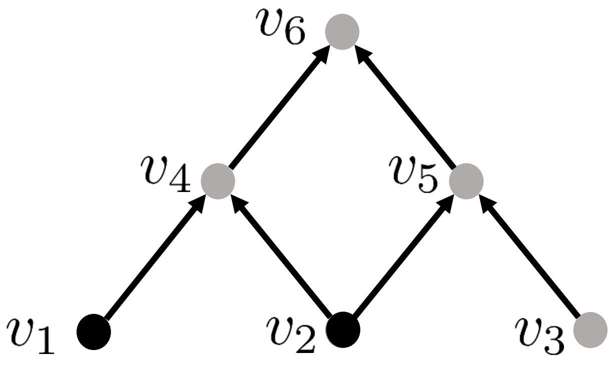

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NS refutation to reversible pebbling

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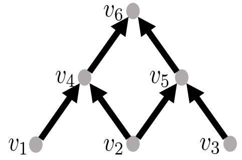

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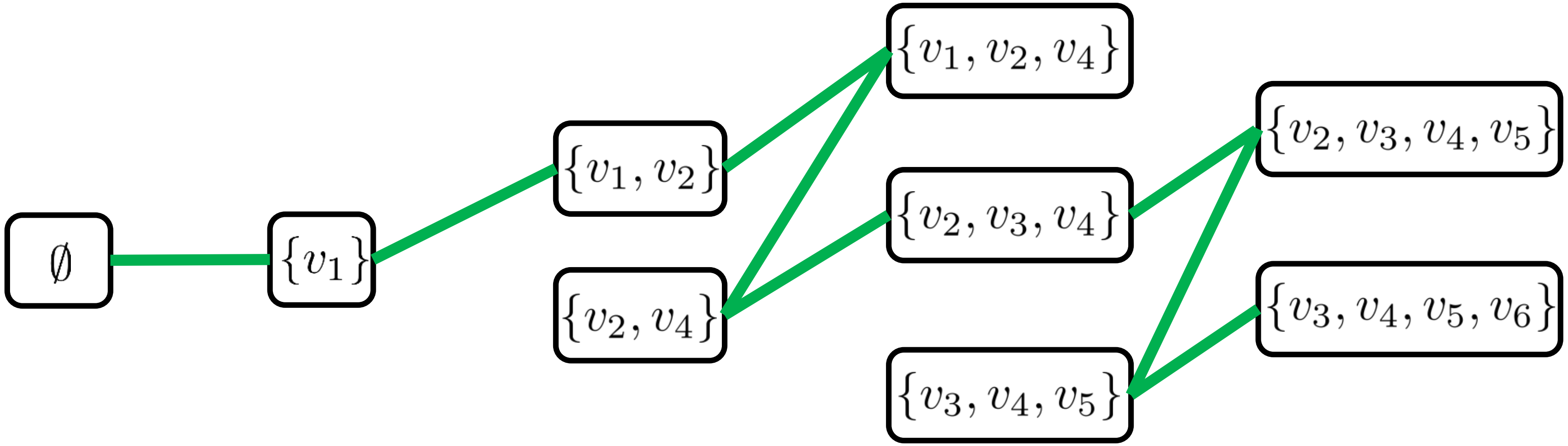

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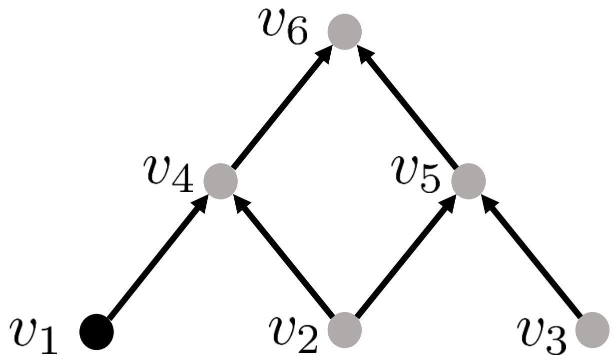

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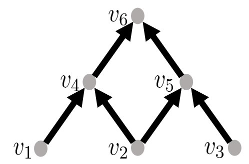

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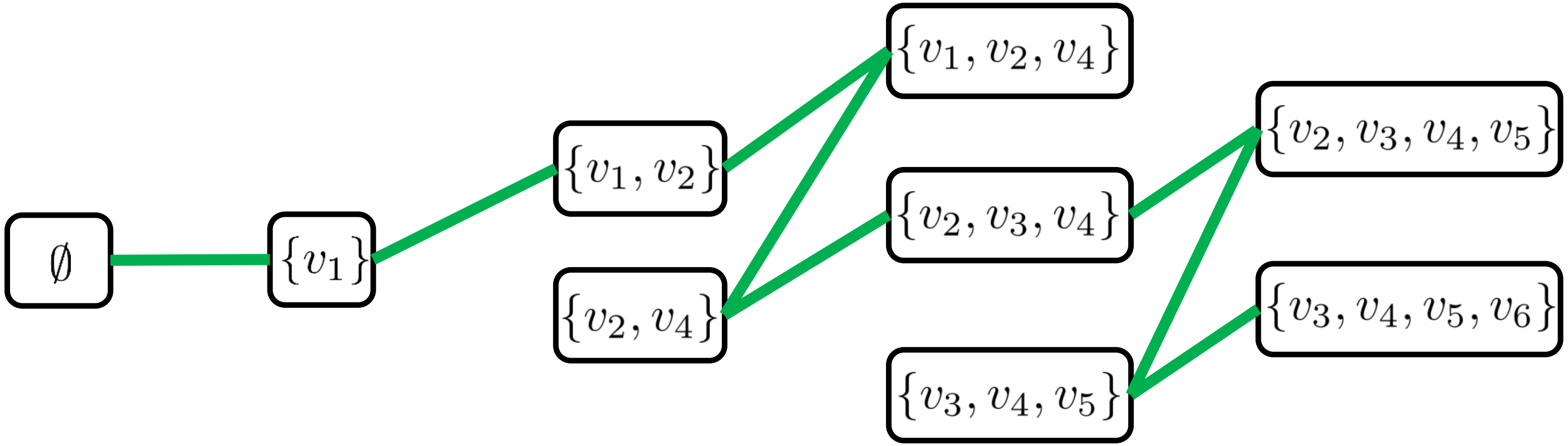

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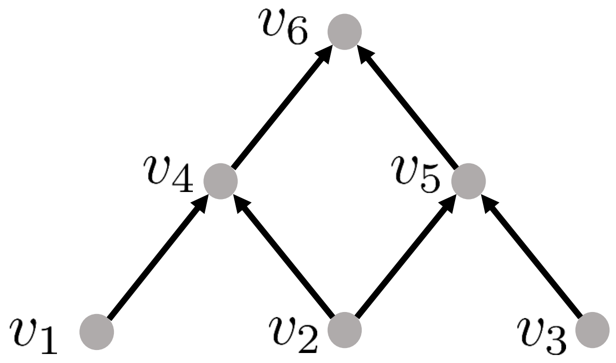

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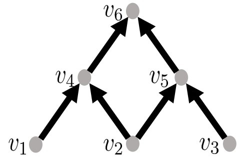

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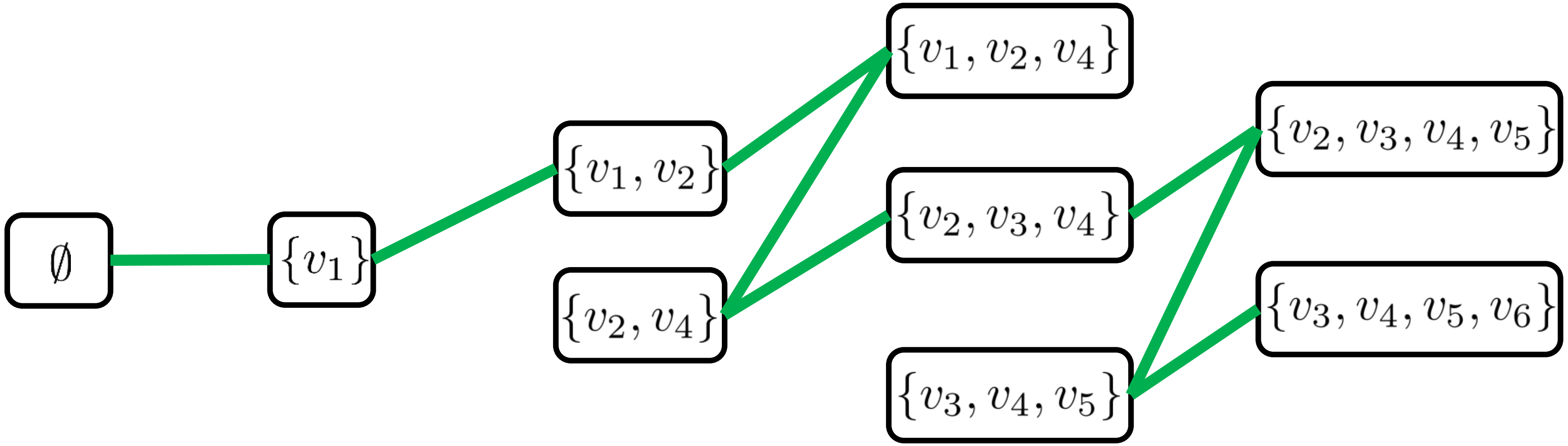

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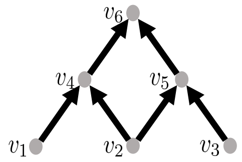

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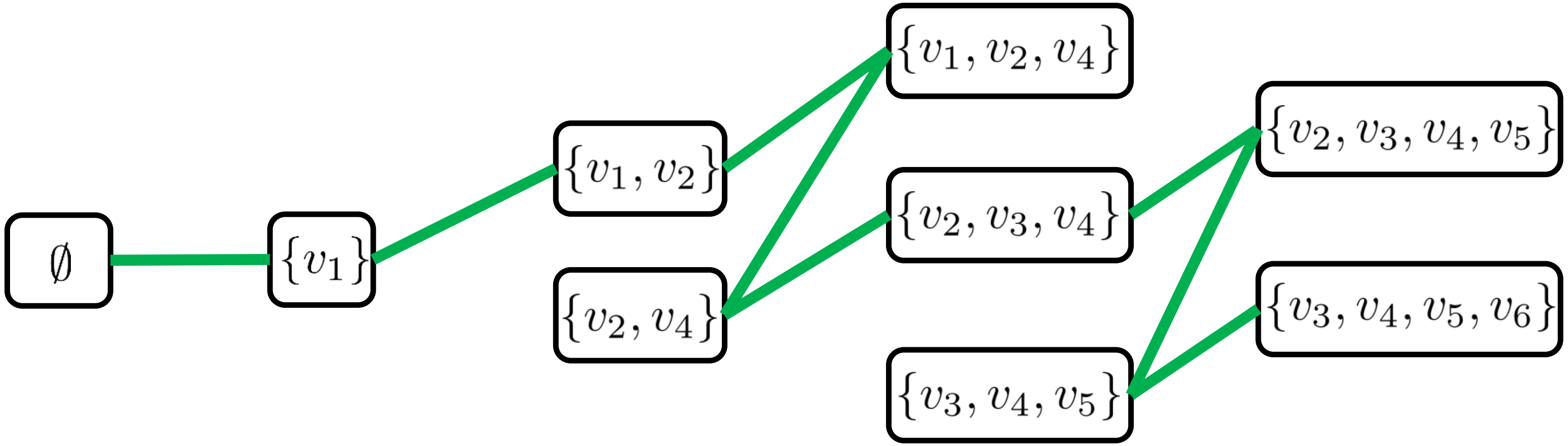

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$\exists$ path from $\emptyset$ to some set containing z

(1) $\deg(\emptyset)$ odd

(2) $z \notin U \neq \emptyset$, $\deg(U)$ even

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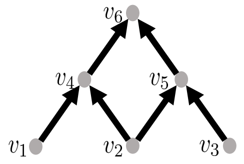

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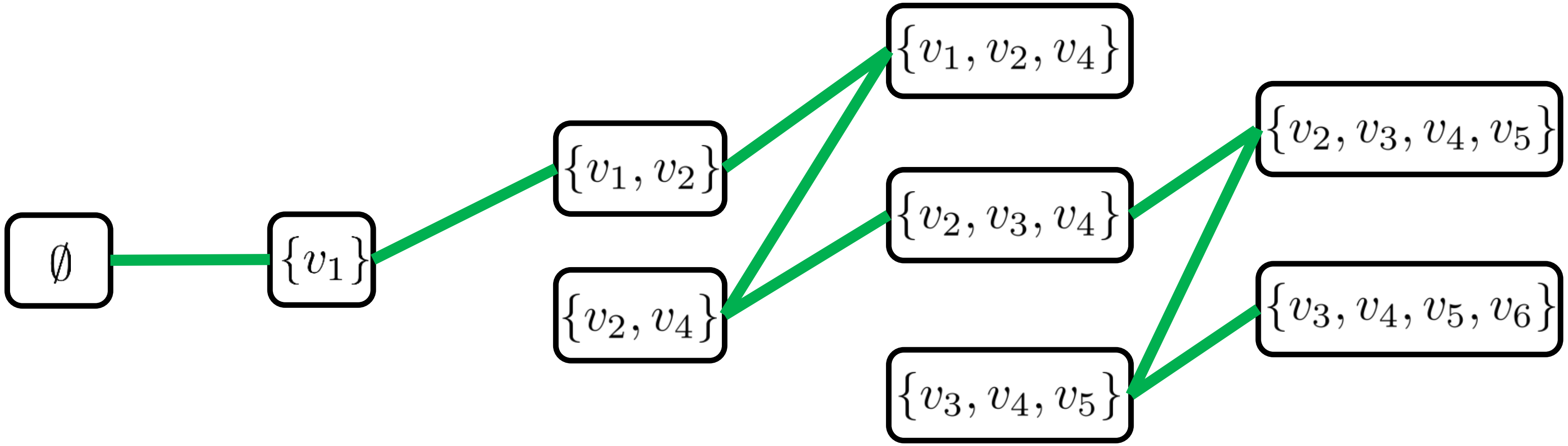

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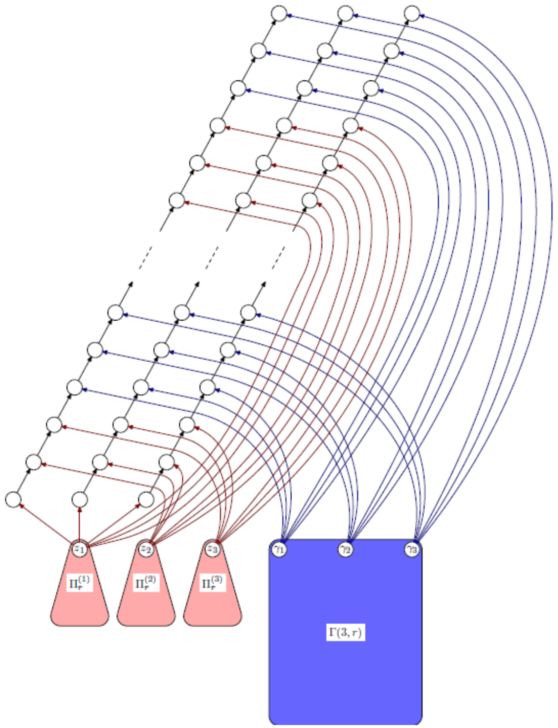
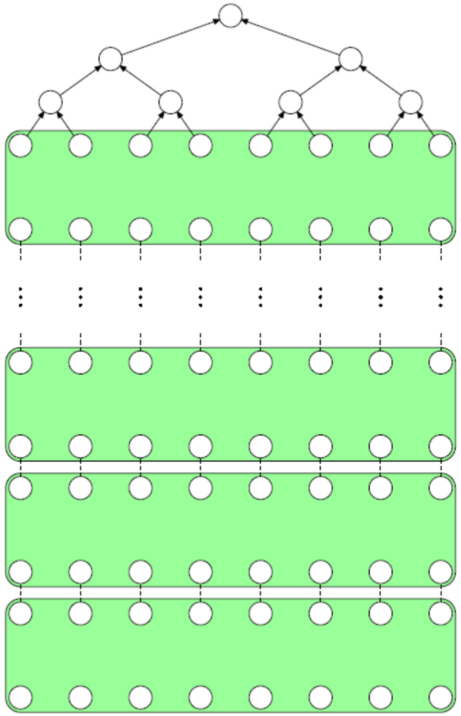
pebbling time $\leq 2 \cdot \#$ edges $= 2 \cdot \frac{\text{NS size}-1}{2} = 2 \cdot 8$

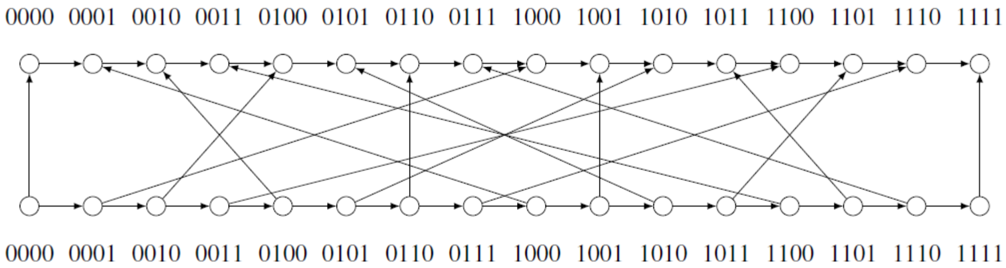
pebbling space $\leq$ NS degree $= 4$

Reversible pebbling time-space trade-offs

- Need reversible pebbling time-space trade-off
- Time-space trade-off has been studied for standard pebbling [CS82, LT82]
- Lower bounds still hold, upper bounds have to be adapted







Concrete example of NS size-degree trade-off

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There is an explicit family of sets of polynomials s.t.

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2. $\exists$ NS refutation in degree $\widetilde{O}(\sqrt[6]{n})$;
3. any NS refutation in degree $\leq \sqrt[3]{n}$ has size $\geq n^{\Omega(\sqrt[6]{n})}$.

Take home

Summing up

- Nullstellensatz refutation $\Leftrightarrow$ reversible pebbling
- Size-degree trade-off for Nullstellensatz

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Thank you!