Short Proofs Are Narrow (Well, Sort of), But Are They Tight?

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Outline of Part I: Proof Complexity and Resolution

Outline of Part II: Resolution Width and Space

Resolution Space
- Definition of Space
- Some Basic Properties

Combinatorial Characterization of Width
- Boolean Existential Pebble Game
- Existential Pebble Game Characterizes Resolution Width

Space is Greater than Width

Open Questions

Part I

Proof Complexity and Resolution

What Is a Proof?

Claim: 25957 is the product of two primes.

True or false? What kind of proof would convince us?

- "I told you so. Just factor and check it yourself!"
  Not much of a proof.
- "$25957 = 101 \cdot 257$. 101 is prime since $101 \equiv 1 \pmod 2$ and $101 \equiv 2 \pmod 3$ and $101 \equiv 1 \pmod 5$ and $101 \equiv 3 \pmod 7$. 257 is prime since $\ldots 257 \equiv 10 \pmod{13}$."
  OK, but maybe even a bit of overkill.
- "$25957 = 101 \cdot 257$; check yourself that these are primes."

Key demand: A proof should be efficiently verifiable.

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Proof system

Proof system for a language L:

Deterministic algorithm $P(s, \pi)$ that runs in time polynomial in $|s|$ and $|\pi|$ such that

- for all $s \in L$ there is a string π (a proof) such that $P(s, \pi) = 1$,
- for all $s \notin L$ it holds for all strings π that $P(s, \pi) = 0$.

Propositional proof system: proof system for the language TAUT of all valid propositional logic formulas (or tautologies)

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Example Propositional Proof System

Example (Truth table)

p	q	r	$(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
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Certainly polynomial-time checkable measured in "proof" size
Why does this not make us happy?

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Proof System Complexity

Complexity $comp_P$ of a proof system P:

Smallest $g : \mathbb{N} \mapsto \mathbb{N}$ such that $s \in L$ if and only if there is a proof π of size $|\pi| \leq g(|s|)$ such that $P(s, \pi) = 1$.

If a proof system is of polynomial complexity, it is said to be polynomially bounded or p-bounded.

Example (Truth table continued)

Truth table is a propositional proof system, but of exponential complexity!

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Proof systems and P vs. NP

Theorem (Cook & Reckhow 1979)

NP = co-NP *if and only if there exists a polynomially bounded propositional proof system.*

Proof.

NP *exactly* the set of languages with *p*-bounded proof systems

$\Rightarrow$ TAUT $\in$ co-NP since F is *not* a tautology iff $\neg F \in$ SAT. If NP = co-NP, then TAUT $\in$ NP has a *p*-bounded proof system by definition.

$\Leftarrow$ Suppose there exists a *p*-bounded proof system. Then TAUT $\in$ NP, and since TAUT is complete for co-NP it follows that NP = co-NP. □

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Polynomial Simulation

The guess is that NP $\neq$ co-NP
Seems that proof of this is lightyears away
(Would imply P $\neq$ NP as a corollary)

Proof complexity tries to approach this distant goal by studying successively stronger propositional proof systems and relating their strengths.

Definition (p-simulation)

P_1 polynomially simulates, or p-simulates, P_2 if there exists a polynomial-time computable function f such that for all $F \in$ TAUT it holds that $P_2(F, \pi) = 1$ iff $P_1(F, f(\pi)) = 1$.

Weak p-simulation: $comp_{P_1} = (comp_{P_2})^{\mathcal{O}(1)}$ but we do not know explicit translation function f from P_2-proofs to P_1-proofs

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Polynomial Equivalence

Definition (p-equivalence)

Two propositional proof systems P_1 and P_2 are polynomially equivalent, or p-equivalent, if each proof system p-simulates the other.

If P_1 p-simulates P_2 but P_2 does not p-simulate P_1, then P_1 is strictly stronger than P_2.

Lots of results proven relating strength of different propositional proof systems

Proof Search Algorithms and Automatizability

But how do we *find* proofs?

Proof search algorithm A_P for propositional proof system P: deterministic algorithm with

- input: formula F
- output: P-proof π of F or report that F is falsifiable

Definition (Automatizability)

P is automatizable if there exists a proof search algorithm A_P such that if $F \in \text{TAUT}$ then A_P on input F outputs a P-proof of F in time polynomial in *the size of a smallest P-proof of F*.

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Short Proofs Seem Hard to Find

Example (Truth table continued)

Truth table is (trivially) an automatizable propositional proof system. (But the proofs we find are of exponential size, so this is not very exciting.)

We want proof systems that are *both*

- strong (i.e., have short proofs for all tautologies) and
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Transforming Tautologies to Unsatisfiable CNFs

Any propositional logic formula F can be converted to formula F' in conjunctive normal form (CNF) such that

- F' only linearly larger than F
- F' unsatisfiable iff F tautology

Idea:

- Introduce new variable x_G for each subformula $G \doteq H_1 \circ H_2$ in F, $\circ \in \{\land, \lor, \rightarrow, \leftrightarrow\}$
- Translate G to set of disjunctive clauses $Cl(G)$ which enforces that the truth value of x_G is computed correctly given truth values of x_{H_1} and x_{H_2}

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Sketch of Transformation

Two examples for $\vee$ and $\rightarrow$ ($\wedge$ and $\leftrightarrow$ are analogous):

$$G \equiv H_1 \vee H_2 : \qquad Cl(G) := \begin{array}{l} (\overline{x}_G \vee x_{H_1} \vee x_{H_2}) \\ \wedge\ (x_G \vee \overline{x}_{H_1}) \\ \wedge\ (x_G \vee \overline{x}_{H_2}) \end{array}$$

$$G \equiv H_1 \rightarrow H_2 : \qquad Cl(G) := \begin{array}{l} (\overline{x}_G \vee \overline{x}_{H_1} \vee x_{H_2}) \\ \wedge\ (x_G \vee x_{H_1}) \\ \wedge\ (x_G \vee \overline{x}_{H_2}) \end{array}$$

- Finally, add clause $\overline{x_F}$

Proof Systems for Refuting Unsatisfiable CNFs

Easy to verify that constructed CNF formula F' is unsatisfiable iff F is a tautology

So any sound and complete proof system which produces refutations of formulas in conjunctive normal form can be used as a propositional proof system

This talk will focus on resolution, which is such a proof system

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \ldots \vee a_k$: set of literals
  At most k literals: k-clause
- CNF formula $F = C_1 \wedge \ldots \wedge C_m$: set of clauses
  k-CNF formula: CNF formula consisting of k-clauses
- $Vars(\cdot)$: set of variables in clause or formula
  $Lit(\cdot)$: set of literals in clause or formula
- $F \vDash D$: semantical implication, $\alpha(F)$ true $\Rightarrow \alpha(D)$ true for all truth value assignments α
- $[n] = \{1, 2, \ldots, n\}$

Resolution Proof System

Resolution derivation $\pi : F \vdash A$ of clause A from F:
Sequence of clauses $\pi = \{D_1, \ldots, D_s\}$ such that $D_s = A$ and each line D_i, $1 \leq i \leq s$, is either

- a clause $C \in F$ (an axiom)
- a resolvent derived from clauses D_j, D_k in π (with $j, k < i$) by the resolution rule

$$\frac{B \vee x \quad C \vee \overline{x}}{B \vee C}$$

resolving on the variable x

Resolution refutation of CNF formula F:
Derivation of empty clause 0 (clause with no literals) from F

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Example Resolution Refutation

$$F = (x \lor z) \land (\overline{z} \lor y) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

Resolution Sound and Complete

Resolution is sound and implicationally complete.

Sound If there is a resolution derivation $\pi : F \vdash A$ then $F \vDash A$

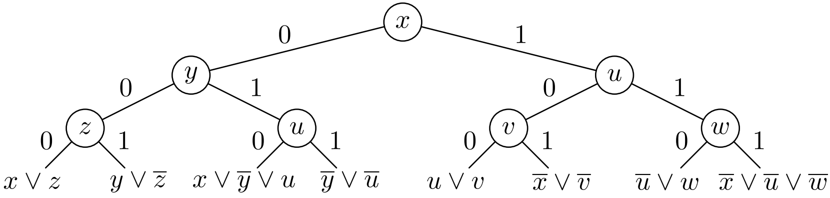
Complete If $F \vDash A$ then there is a resolution derivation $\pi : F \vdash A'$ for some $A' \subseteq A$.

In particular,

$$F \text{ is unsatisfiable} \Leftrightarrow \exists \text{ resolution refutation of } F$$

Completeness of Resolution: Proof by Example

Decision tree:



Resulting resolution refutation:

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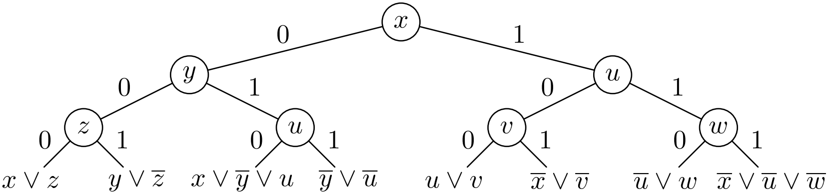

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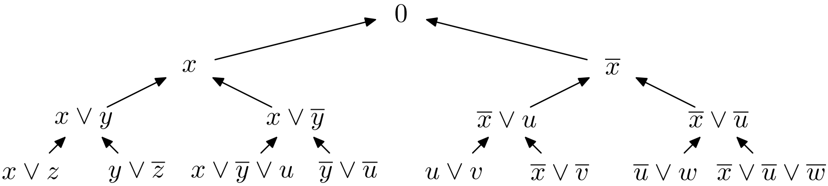

Derivation Graph and Tree-Like Derivations

Derivation graph G_π of a resolution derivation π:
directed acyclic graph (DAG) with

- vertices: clauses of the derivations
- edges: from $B \lor x$ and $C \lor \overline{x}$ to $B \lor C$ for each application of the resolution rule

A resolution derivation π is tree-like if G_π is a tree
(We can make copies of axiom clauses to make G_π into a tree)

Example

Our example resolution proof is tree-like.
(The derivation graph is on the previous slide.)

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Length

- Length $L(F)$ of CNF formula F is # clauses in it
- Length of derivation $\pi : F \vdash A$ is # clauses in π (with repetitions)
- Length of deriving A from F is

$$L(F \vdash A) = \min_{\pi : F \vdash A} \{L(\pi)\}$$

  where minimum taken over all derivations of A
- Length of deriving A from F in *tree-like resolution* is $L_T(F \vdash A)$ (min of all tree-like derivations)

1. $x \lor z$
2. $\overline{z} \lor y$
3. $x \lor \overline{y} \lor u$
4. $\overline{y} \lor \overline{u}$
5. $u \lor v$
6. $\overline{x} \lor \overline{v}$
7. $\overline{u} \lor w$
8. $\overline{x} \lor \overline{u} \lor \overline{w}$
9. $x \lor y$
10. $x \lor \overline{y}$
11. $\overline{x} \lor u$
12. $\overline{x} \lor \overline{u}$
13. x
14. $\overline{x}$
15. 0

Length 15

Exponential Lower Bound for Proof Length

Theorem (Haken 1985)

There is a family of unsatisfiable CNF formulas $\{F_n\}_{n=1}^{\infty}$ *of size polynomial in n such that* $L(F_n \vdash 0) = \exp(\Omega(n))$.

Also known: general resolution is exponentially stronger than tree-like resolution (Bonet et al. 1998, Ben-Sasson et al. 1999)

Resolution widely used in practice anyway because of nice properties for proof search algorithms (but is probably not automatizable)

Theoretical point of view: we want to understand resolution
Gain insights and develop techniques that perhaps can be used to attack more powerful proof systems

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Weakening

In proofs, sometimes convenient to add a derivation rule for weakening

$$\frac{B}{B \vee C}$$

(for arbitrary clauses B, C).

Proposition

Any resolution refutation $\pi : F \vdash 0$ using weakening can be transformed into a refutation $\pi' : F \vdash 0$ without weakening in at most the same length.

Proof.

Easy proof by induction over the resolution refutation. □

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Restriction

Restriction ρ: partial truth value assignment
Represented as set of literals $\rho = \{a_1, \ldots, a_m\}$ set to true by ρ

For a clause C, the ρ-restriction of C is

$$C|_\rho = \begin{cases} 1 & \text{if } \rho \cap Lit(C) \neq \emptyset \\ C \setminus \{\overline{a} \mid a \in \rho\} & \text{otherwise} \end{cases}$$

where 1 denotes the trivially true clause

For a formula F, define $F|_\rho = \bigwedge_{C \in F} C|_\rho$

For a derivation $\pi = \{D_1, \ldots, D_s\}$, define $\pi|_\rho = \{D_1|_\rho, \ldots, D_s|_\rho\}$ (with all trivial clauses 1 removed)

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Example Restriction

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2.	$\overline{z} \vee y$	Axiom in F
3.	$x \vee \overline{y} \vee u$	Axiom in F
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14.	$\overline{x}$	Res$(11, 12)$
15.	0	Res$(13, 14)$

$\pi|_x =$

1.	—	
2.	$\overline{z} \vee y$	Axiom in $F|_x$
3.	—	
4.	$\overline{y} \vee \overline{u}$	Axiom in $F|_x$
5.	$u \vee v$	Axiom in $F|_x$
6.	$\overline{v}$	Axiom in $F|_x$
7.	$\overline{u} \vee w$	Axiom in $F|_x$
8.	$\overline{u} \vee \overline{w}$	Axiom in $F|_x$
9.	—	
10.	—	
11.	u	Res$(5, 6)$
12.	$\overline{u}$	Res$(7, 8)$
13.	—	
14.	0	Res$(11, 12)$

Restrictions Preserve Resolution Derivations

Proposition

If $\pi : F \vdash A$ is a resolution derivation and ρ is a restriction on $Vars(F)$, then $\pi|_\rho$ is a derivation of $A|_\rho$ from $F|_\rho$, possibly using weakening.

Proof.

Easy proof by induction over the resolution derivation. □

In particular, if $\pi : F \vdash 0$ then $\pi|_\rho$ can be transformed into a resolution refutation of $F|_\rho$ *without weakening* in at most the same length as π.

Width

- Width $W(C)$ of clause C is $|C|$, i.e., # literals
- Width of formula F or derivation π is width of the widest clause in the formula / derivation
- Width of deriving A from F is

$$W(F \vdash A) = \min_{\pi : F \vdash A} \{ W(\pi) \}$$

(No difference between tree-like and general resolution)

Always $W(F \vdash 0) \leq |Vars(F)|$

1. $x \lor z$
2. $\overline{z} \lor y$
3. $x \lor \overline{y} \lor u$
4. $\overline{y} \lor \overline{u}$
5. $u \lor v$
6. $\overline{x} \lor \overline{v}$
7. $\overline{u} \lor w$
8. $\overline{x} \lor \overline{u} \lor \overline{w}$
9. $x \lor y$
10. $x \lor \overline{y}$
11. $\overline{x} \lor u$
12. $\overline{x} \lor \overline{u}$
13. x
14. $\overline{x}$
15. 0

Width 3

Width and Length

A narrow resolution proof is necessarily short.

For a proof in width w, $\left(2 \cdot |Vars(F)|\right)^{w}$ is an upper bound on the number of possible clauses.

Ben-Sasson & Wigderson proved (sort of) that the converse also holds.

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Technical Lemma 1

Lemma

If $W(F|_x \vdash A) \leq w$ then $W(F \vdash A \lor \overline{x}) \leq w + 1$ (possibly by use of the weakening rule).

Proof.

- Suppose $\pi = \{D_1, \ldots, D_s\}$ derives A from $F|_x$ in width $W(\pi) \leq w$.
- Add the literal $\overline{x}$ to all clauses in π.
- Claim: this yields a legal derivation π' from F (possibly with weakening).
- If so, obviously $W(\pi') \leq w + 1$, and last line is $A \lor \overline{x}$. □

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Proof of Technical Lemma 1 (continued)

Proof of claim.

Need to show that each $D_i \vee \overline{x} \in \pi'$ can be derived from previous clauses by resolution and/or weakening.

Let $F_{\overline{x}} = \{C \in F \mid \overline{x} \in Lit(C)\}$ be the set of all clauses of F containing the literal $\overline{x}$.

Three cases:

1. $D_i \in F_{\overline{x}}|_x$: This means that $D_i \vee \overline{x} \in F$, which is OK.
2. $D_i \in F|_x \setminus F_{\overline{x}}|_x$: This means that $D_i \in F$, so $D_i \vee \overline{x}$ can be derived by weakening.
3. D_i derived from $D_j, D_k \in \pi$ by resolution: By induction $D_j \vee \overline{x}$ and $D_k \vee \overline{x} \in \pi'$ derivable; resolve to get $D_i \vee \overline{x}$. □

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Lemma

If

- $W(F|_x \vdash 0) \leq w - 1$ *and*
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then

- $W(F \vdash 0) \leq \max\{w, W(F)\}$.

Proof.

- Derive $\overline{x}$ in width $\leq w$ by Technical Lemma 1.
- Resolve $\overline{x}$ with all clauses $C \in F$ containing literal x to get $F|_{\overline{x}}$ in width $\leq W(F)$.
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Warm-Up: Tree-Like Resolution

Theorem (Ben-Sasson & Wigderson 1999)

For tree-like resolution, the width of refuting a CNF formula F is bounded from above by

$$W(F \vdash 0) \leq W(F) + \log_2 L_T(F \vdash 0).$$

Corollary

For tree-like resolution, the length of refuting a CNF formula F is bounded from below by

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Proof for Tree-Like Resolution (1 / 2)

Proof by nested induction over b and # variables n that

$$L_T(F \vdash 0) \leq 2^b \;\Rightarrow\; W(F \vdash 0) \leq W(F) + b$$

Base cases:

$b = 0 \Rightarrow$ proof of length 1 $\Rightarrow$ empty clause $0 \in F$

$n = 1 \Rightarrow$ formula over 1 variable, i.e., $x \wedge \overline{x} \Rightarrow \exists$ proof of width 1

Induction step:

Suppose for formula F with n variables that π is tree-like refutation in length $\leq 2^b$

Last step in refutation $\pi : F \vdash 0$ is $\frac{x \quad \overline{x}}{0}$ for some x

Let π_x and $\pi_{\overline{x}}$ be the tree-like subderivations of x and $\overline{x}$, respectively

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Since $L(\pi) = L(\pi_x) + L(\pi_{\overline{x}}) + 1 \leq 2^b$
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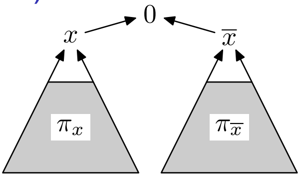
Suppose w.l.o.g. $L(\pi_{\overline{x}}) \leq 2^{b-1}$



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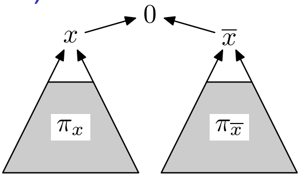

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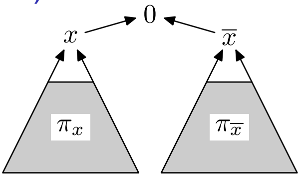

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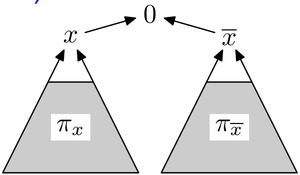

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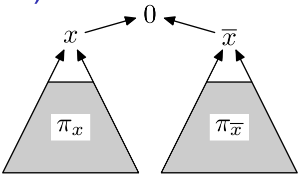

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The General Case

Theorem (Ben-Sasson & Wigderson 1999)

The width of refuting a CNF formula F over n variables in general resolution is bounded from above by

$$W(F \vdash 0) \leq W(F) + \mathcal{O}\left(\sqrt{n \log L(F \vdash 0)}\right).$$

Note: $2^{n+1} - 1$ maximal possible proof length, so bound is

$$W(F \vdash 0) \lessapprox W(F) + \sqrt{\log(\text{max possible}) \cdot \log L(F \vdash 0)}$$

This bound on width in terms of length is essentially optimal (Bonet & Galesi 1999).

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The General Case: Corollary

Corollary

For general resolution, the length of refuting a CNF formula F over n variables is bounded from below by

$$L(F \vdash 0) \geq \exp\left(\Omega\left(\frac{(W(F \vdash 0) - W(F))^2}{n}\right)\right).$$

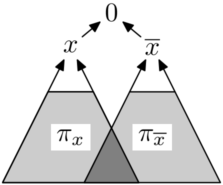
Has been used to simplify many length lower bound proofs in resolution (and to prove a couple of new ones)

Need $W(F \vdash 0) - W(F) = \omega(\sqrt{n})$ to get non-trivial bounds

(Not a) Proof of the General Case

Proof for tree-like resolution breaks down in general case

Not true that $L(\pi) = L(\pi_x) + L(\pi_{\overline{x}}) + 1$
Subderivations π_x and $\pi_{\overline{x}}$ may share clauses!



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- Look at very wide clauses in π
- Eliminate many of them by applying restriction setting commonly occurring literal to true
- More complicated inductive argument (still exponential blow-up in length)

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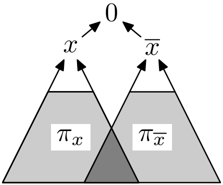

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Part II

Resolution Width and Space

Outline of Part II: Resolution Width and Space

Introducing Space

- Results on width lead to question: Can other complexity measures yield interesting insights as well?
- Esteban & Torán (1999) introduced proof space (maximal # clauses in memory while verifying proof)
- Many lower bounds for space proven
  All turned out to match width bounds!
  Coincidence?
- Atserias & Dalmau (2003): space $\geq$ width $-$ constant for k-CNF formulas

The subject of the 2nd part of this talk

Resolution Derivation (Revisited)

Sequence of sets of clauses, or clause configurations, $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for clause $C \in F$ (axiom)

Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for clause $C \in \mathbb{C}_{t-1}$

Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C \vee D\}$ for clause $C \vee D$ inferred by resolution rule from $C \vee x, D \vee \overline{x} \in \mathbb{C}_{t-1}$

Resolution derivation $\pi : F \vdash D$ of clause D from F:
Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_\tau = \{D\}$

Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F

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Sequence of sets of clauses, or clause configurations, $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

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Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for clause $C \in \mathbb{C}_{t-1}$

Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C \vee D\}$ for clause $C \vee D$ inferred by resolution rule from $C \vee x, D \vee \overline{x} \in \mathbb{C}_{t-1}$

Resolution derivation $\pi : F \vdash D$ of clause D from F:
Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_\tau = \{D\}$

Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Resolution Derivation (Revisited)

Sequence of sets of clauses, or clause configurations, $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

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Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Resolution Derivation (Revisited)

Sequence of sets of clauses, or clause configurations, $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

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Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C \lor D\}$ for clause $C \lor D$ inferred by resolution rule from $C \lor x, D \lor \overline{x} \in \mathbb{C}_{t-1}$

Resolution derivation $\pi : F \vdash D$ of clause D from F:
Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_\tau = \{D\}$

Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Resolution Derivation (Revisited)

Sequence of sets of clauses, or clause configurations, $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for clause $C \in F$ (axiom)

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Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C \vee D\}$ for clause $C \vee D$ inferred by resolution rule from $C \vee x, D \vee \overline{x} \in \mathbb{C}_{t-1}$

Resolution derivation $\pi : F \vdash D$ of clause D from F:
Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_\tau = \{D\}$

Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Resolution Derivation (Revisited)

Sequence of sets of clauses, or clause configurations, $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for clause $C \in F$ (axiom)

Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for clause $C \in \mathbb{C}_{t-1}$

Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C \vee D\}$ for clause $C \vee D$ inferred by resolution rule from $C \vee x, D \vee \overline{x} \in \mathbb{C}_{t-1}$

Resolution derivation $\pi : F \vdash D$ of clause D from F:
Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_\tau = \{D\}$

Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom
2.	$\overline{z} \lor y$	Axiom
3.	$x \lor \overline{y} \lor u$	Axiom
4.	$\overline{y} \lor \overline{u}$	Axiom
5.	$u \lor v$	Axiom
6.	$\overline{x} \lor \overline{v}$	Axiom
7.	$\overline{u} \lor w$	Axiom
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom
9.	$x \lor y$	$\mathrm{Res}(1,2)$
10.	$x \lor \overline{y}$	$\mathrm{Res}(3,4)$
11.	$\overline{x} \lor u$	$\mathrm{Res}(5,6)$
12.	$\overline{x} \lor \overline{u}$	$\mathrm{Res}(7,8)$
13.	x	$\mathrm{Res}(9,10)$
14.	$\overline{x}$	$\mathrm{Res}(11,12)$
15.	0	$\mathrm{Res}(13,14)$

$$\left[\quad \right]$$

Empty start configuration

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1,2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3,4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5,6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7,8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9,10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11,12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	Res$(13,14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor z \\ \\ \\ \\ \end{bmatrix}$$

Download axiom $x \lor z$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor z \\ \\ \\ \\ \end{bmatrix}$$

Download axiom $x \lor z$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor z \\ \overline{z} \lor y \\ \\ \\ \end{bmatrix}$$

Download axiom $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor z \\ \overline{z} \lor y \\ \\ \\ \end{bmatrix}$$

Download axiom $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor z \\ \overline{z} \lor y \\ \\ \\ \end{bmatrix}$$

Infer $x \lor y$ from $x \lor z$ and $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor z \\ \overline{z} \lor y \\ x \lor y \\ \\ \end{bmatrix}$$

Infer $x \lor y$ from $x \lor z$ and $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor z \\ \overline{z} \lor y \\ x \lor y \\ \\ \end{bmatrix}$$

Infer $x \lor y$ from
$x \lor z$ and $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\left[\begin{array}{l} x \lor z \\ \overline{z} \lor y \\ x \lor y \\ \\ \end{array}\right]$$

Erase clause $x \lor z$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} \overline{z} \lor y \\ x \lor y \\ \\ \\ \end{bmatrix}$$

Erase clause $x \lor z$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} \overline{z} \lor y \\ x \lor y \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \lor u \\ \\ \\ \end{bmatrix}$$

Download axiom $x \lor \overline{y} \lor u$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \lor u \\ \\ \\ \end{bmatrix}$$

Download axiom $x \lor \overline{y} \lor u$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res(3, 4)
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res(5, 6)
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
5.	$u \vee v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \vee w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

$$\begin{bmatrix} x \vee y \\ x \vee \overline{y} \vee u \\ \overline{y} \vee \overline{u} \\ \\ \end{bmatrix}$$

Download axiom $\overline{y} \vee \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \lor u \\ \overline{y} \lor \overline{u} \\ \\ \end{bmatrix}$$

Download axiom $\overline{y} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\left[\begin{array}{c} x \lor y \\ x \lor \overline{y} \lor u \\ \overline{y} \lor \overline{u} \\ \\ \end{array}\right]$$

Infer $x \lor \overline{y}$ from
$x \lor \overline{y} \lor u$ and $\overline{y} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \lor u \\ \overline{y} \lor \overline{u} \\ x \lor \overline{y} \\ \end{bmatrix}$$

Infer $x \lor \overline{y}$ from
$x \lor \overline{y} \lor u$ and $\overline{y} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	$\mathrm{Res}(1,2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	$\mathrm{Res}(3,4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	$\mathrm{Res}(5,6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	$\mathrm{Res}(7,8)$
5.	$u \lor v$	Axiom	13.	x	$\mathrm{Res}(9,10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	$\mathrm{Res}(11,12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	$\mathrm{Res}(13,14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \lor u \\ \overline{y} \lor \overline{u} \\ x \lor \overline{y} \\ \end{bmatrix}$$

Infer $x \lor \overline{y}$ from
$x \lor \overline{y} \lor u$ and $\overline{y} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom
2.	$\overline{z} \lor y$	Axiom
3.	$x \lor \overline{y} \lor u$	Axiom
4.	$\overline{y} \lor \overline{u}$	Axiom
5.	$u \lor v$	Axiom
6.	$\overline{x} \lor \overline{v}$	Axiom
7.	$\overline{u} \lor w$	Axiom
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom
9.	$x \lor y$	Res(1, 2)
10.	$x \lor \overline{y}$	Res(3, 4)
11.	$\overline{x} \lor u$	Res(5, 6)
12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
13.	x	Res(9, 10)
14.	$\overline{x}$	Res(11, 12)
15.	0	Res(13, 14)

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \lor u \\ \overline{y} \lor \overline{u} \\ x \lor \overline{y} \end{bmatrix}$$

Erase clause $x \lor \overline{y} \lor u$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\left[\begin{array}{c} x \lor y \\ \overline{y} \lor \overline{u} \\ x \lor \overline{y} \\ \\ \end{array}\right]$$

Erase clause $x \lor \overline{y} \lor u$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ \overline{y} \lor \overline{u} \\ x \lor \overline{y} \\ \\ \end{bmatrix}$$

Erase clause $\overline{y} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{y} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \\ \\ \\ \end{bmatrix}$$

Infer x from $x \lor y$ and $x \lor \overline{y}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \\ x \\ \\ \end{bmatrix}$$

Infer x from
$x \lor y$ and $x \lor \overline{y}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \\ x \\ \\ \end{bmatrix}$$

Infer x from
$x \lor y$ and $x \lor \overline{y}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1, 2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3, 4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5, 6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7, 8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	Res$(13, 14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor y \\ x \lor \overline{y} \\ x \\ \\ \end{bmatrix}$$

Erase clause $x \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor \overline{y} \\ x \\ \\ \\ \end{bmatrix}$$

Erase clause $x \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \lor \overline{y} \\ x \\ \\ \\ \end{bmatrix}$$

Erase clause $x \lor \overline{y}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \\ \\ \\ \end{bmatrix}$$

Erase clause $x \lor \overline{y}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ u \lor v \\ \\ \\ \end{bmatrix}$$

Download axiom $u \lor v$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ u \lor v \\ \\ \\ \end{bmatrix}$$

Download axiom $u \lor v$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ u \lor v \\ \overline{x} \lor \overline{v} \\ \\ \end{bmatrix}$$

Download axiom $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ u \lor v \\ \overline{x} \lor \overline{v} \\ \\ \end{bmatrix}$$

Download axiom $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ u \lor v \\ \overline{x} \lor \overline{v} \\ \\ \end{bmatrix}$$

Infer $\overline{x} \lor u$ from $u \lor v$ and $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ u \lor v \\ \overline{x} \lor \overline{v} \\ \overline{x} \lor u \end{bmatrix}$$

Infer $\overline{x} \lor u$ from $u \lor v$ and $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1, 2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3, 4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5, 6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7, 8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	Res$(13, 14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ u \lor v \\ \overline{x} \lor \overline{v} \\ \overline{x} \lor u \\ \end{bmatrix}$$

Infer $\overline{x} \lor u$ from $u \lor v$ and $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ u \lor v \\ \overline{x} \lor \overline{v} \\ \overline{x} \lor u \\ \end{bmatrix}$$

Erase clause $u \lor v$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1,2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3,4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5,6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7,8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9,10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11,12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	Res$(13,14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor \overline{v} \\ \overline{x} \lor u \\ \\ \end{bmatrix}$$

Erase clause $u \lor v$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor \overline{v} \\ \overline{x} \lor u \\ \\ \end{bmatrix}$$

Erase clause $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1, 2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3, 4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5, 6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7, 8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	Res$(13, 14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{u} \lor w \\ \\ \end{bmatrix}$$

Download axiom $\overline{u} \lor w$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{u} \lor w \\ \\ \end{bmatrix}$$

Download axiom $\overline{u} \lor w$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{u} \lor w \\ \overline{x} \lor \overline{u} \lor \overline{w} \\ \end{bmatrix}$$

Download axiom $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1,2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3,4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5,6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7,8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9,10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11,12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	Res$(13,14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{u} \lor w \\ \overline{x} \lor \overline{u} \lor \overline{w} \\ \end{bmatrix}$$

Download axiom $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{u} \lor w \\ \overline{x} \lor \overline{u} \lor \overline{w} \\ \end{bmatrix}$$

Infer $\overline{x} \lor \overline{u}$ from $\overline{u} \lor w$ and $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{u} \lor w \\ \overline{x} \lor \overline{u} \lor \overline{w} \\ \overline{x} \lor \overline{u} \end{bmatrix}$$

Infer $\overline{x} \lor \overline{u}$ from $\overline{u} \lor w$ and $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{u} \lor w \\ \overline{x} \lor \overline{u} \lor \overline{w} \\ \overline{x} \lor \overline{u} \end{bmatrix}$$

Infer $\overline{x} \lor \overline{u}$ from $\overline{u} \lor w$ and $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{u} \lor w \\ \overline{x} \lor \overline{u} \lor \overline{w} \\ \overline{x} \lor \overline{u} \end{bmatrix}$$

Erase clause $\overline{u} \lor w$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{x} \lor \overline{u} \lor \overline{w} \\ \overline{x} \lor \overline{u} \\ \\ \end{bmatrix}$$

Erase clause $\overline{u} \lor w$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{x} \lor \overline{u} \lor \overline{w} \\ \overline{x} \lor \overline{u} \\ \\ \end{bmatrix}$$

Erase clause $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1, 2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3, 4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5, 6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7, 8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	Res$(13, 14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{x} \lor \overline{u} \\ \\ \end{bmatrix}$$

Erase clause $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{x} \lor \overline{u} \\ \\ \end{bmatrix}$$

Infer $\overline{x}$ from
$\overline{x} \lor u$ and $\overline{x} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{x} \lor \overline{u} \\ \overline{x} \\ \end{bmatrix}$$

Infer $\overline{x}$ from
$\overline{x} \lor u$ and $\overline{x} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res$(1, 2)$
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res$(3, 4)$
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res$(5, 6)$
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res$(7, 8)$
5.	$u \vee v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \vee w$	Axiom	15.	0	Res$(13, 14)$
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \vee u \\ \overline{x} \vee \overline{u} \\ \overline{x} \\ \ \end{bmatrix}$$

Infer $\overline{x}$ from
$\overline{x} \vee u$ and $\overline{x} \vee \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1, 2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3, 4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5, 6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7, 8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	Res$(13, 14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor u \\ \overline{x} \lor \overline{u} \\ \overline{x} \\ \end{bmatrix}$$

Erase clause $\overline{x} \lor u$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\left[\begin{array}{l} x \\ \overline{x} \lor \overline{u} \\ \overline{x} \\ \\ \end{array}\right]$$

Erase clause $\overline{x} \lor u$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \lor \overline{u} \\ \overline{x} \\ \\ \end{bmatrix}$$

Erase clause $\overline{x} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{x} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \\ \\ \\ \end{bmatrix}$$

Infer 0 from
x and $\overline{x}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1, 2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3, 4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5, 6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7, 8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \lor w$	Axiom	15.	0	Res$(13, 14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \\ 0 \\ \\ \end{bmatrix}$$

Infer 0 from
x and $\overline{x}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \\ 0 \\ \\ \end{bmatrix}$$

Infer 0 from x and $\overline{x}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} x \\ \overline{x} \\ 0 \\ \\ \end{bmatrix}$$

Erase clause x

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} \overline{x} \\ 0 \\ \\ \\ \end{bmatrix}$$

Erase clause x

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$$\begin{bmatrix} \overline{x} \\ 0 \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{x}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res(3, 4)
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res(5, 6)
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
5.	$u \vee v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \vee w$	Axiom	15.	0	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

$$\begin{bmatrix} 0 \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{x}$

Space

- Space of resolution derivation $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ is max # clauses in any configuration

$$Sp(\pi) = \max_{t \in [\tau]} \{ |\mathbb{C}_t| \}$$

- Space of deriving D from F is

$$Sp(F \vdash D) = \min_{\pi : F \vdash D} \{ Sp(\pi) \}$$

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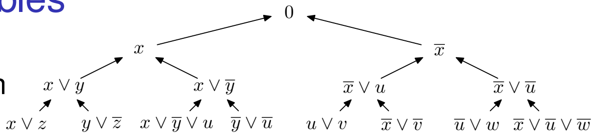
Space $\lessapprox$ # variables

Consider decision tree for F



n variables $\Rightarrow$ height of decision tree at most n

By induction:
Clause at root of subtree of height h derivable in space $h + 2$

- Derive left child clause in space $h + 1$ and keep in memory
- Derive right child clause in space $1 + (h + 1)$
- Resolve the two children clauses to get root clause

Theorem
$Sp(F \vdash 0) \leq |Vars(F)| + 2$

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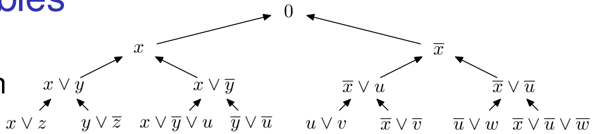

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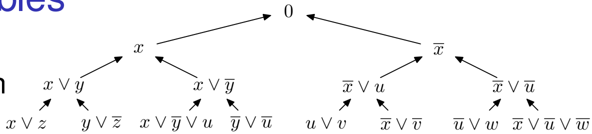

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Minimally Unsatisfiable CNF formula

Definition

An unsatisfiable CNF formula F is minimally unsatisfiable if removing any clause from F makes it satisfiable.

Example

$$F = (x \vee z) \wedge (\overline{z} \vee y) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u}) \wedge (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w})$$

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Min Unsat CNFs Have More Clauses than Variables

Lemma

Any minimally unsatisfiable CNF formula must have more clauses than variables.

Proof.

- Consider bipartite graph on $F \times Vars(F)$ with edges from clauses to variables occurring in the clauses
- No matching, so by Hall's theorem $\exists\ G \subseteq F$ such that $|G| > |N(G)|$ (where $N(\cdot)$ is the set of neighbours)
- Pick G of max size. Suppose $G \neq F$. Then G is satisfiable.
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$Sp(F \vdash 0) \leq L(F) + 1$

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Upper Bounds in # Clauses and # Variables Tight

We just showed

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Thus the interesting question is which formulas demand this much space, and which formulas can be refuted in e.g. logarithmic or even constant space.

Theorem (Alekhnovich et al. 2000, Torán 1999)

There is a polynomial-size family $\{F_n\}_{n=1}^{\infty}$ of unsatisfiable 3-CNF formulas such that $Sp(F \vdash 0) = \Omega(L(F)) = \Omega(|Vars(F)|)$.

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Informal Description of Existential Pebble Game

Game between Spoiler and Duplicator over CNF formula F
Duplicator claims formula is satisfiable
Spoiler wants to disprove this, but suffers from light senility (can only keep p variable assignments in memory)

In each round, Spoiler

- picks a variable to which Duplicator must assign a value, or
- forgets a variable (can choose which)

In each round, Duplicator

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Formal Definition

Duplicator wins the Boolean existential *p*-pebble game over the CNF formula *F* if there is a nonempty family $\mathcal{H}$ of partial truth value assignments that do not falsify any clause in *F* and for which the following holds:

1. If $\alpha \in \mathcal{H}$ then $|\alpha| \leq p$.
2. If $\alpha \in \mathcal{H}$ and $\beta \subseteq \alpha$ then $\beta \in \mathcal{H}$.
3. If $\alpha \in \mathcal{H}$, $|\alpha| < p$ and $x \in Vars(F)$ then there exists a $\beta \in \mathcal{H}$ such that $\alpha \subseteq \beta$ and x is in the domain of β.

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Constructive Strategies

If there is a winning strategy for Duplicator, then there is a deterministic winning strategy that for each $\alpha \in \mathcal{H}$ and each move of Spoiler defines a move β for Duplicator.

Proposition

If Duplicator has no winning strategy, then there is a winning strategy (in the form of a partial function from partial truth value assignments to variable queries/deletions) for Spoiler.

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The number of possible deterministic strategies for Duplicator is finite, so Spoiler can build a strategy by evaluating all possible responses to sequences of queries and deletions. □

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Existential Pebble Game Characterizes Width

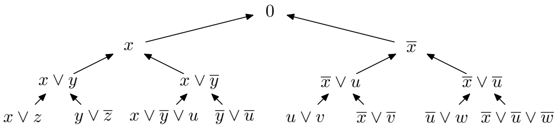
It turns out that the Boolean existential p-pebble game exactly characterizes resolution width.

Theorem (Atserias & Dalmau 2003)

The CNF formula F has a resolution refutation of width $\leq p$ if and only if Spoiler wins the existential $(p+1)$-pebble game on F.

Narrow Proof Yields Winning Strategy for Spoiler

- Given $\pi : F \vdash 0$ with DAG G_π.



- Spoiler starts at the vertex for 0 and inductively queries the variable resolved upon to to get there
- Spoiler moves to the assumption clause D falsified by Duplicator's answer and forgets all variables not in D
- Repeat for the new clause et cetera
- Sooner or later Spoiler reaches a falsified axiom, having used no more than $W(\pi) + 1$ variables simultaneously ($+1$ is for the variable resolved on)

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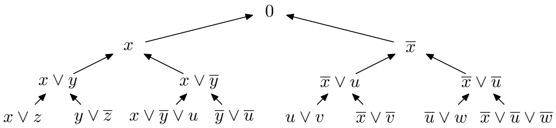

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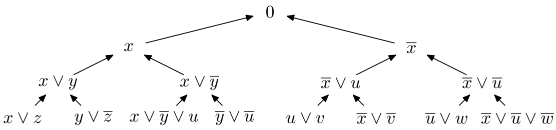

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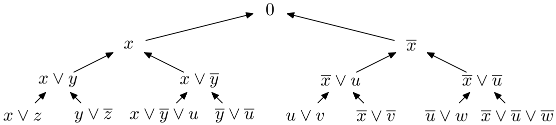

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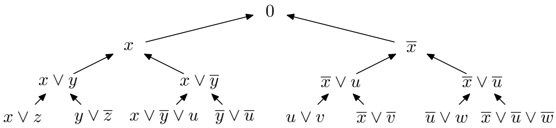

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Given strategy for Spoiler, build DAG G_π as follows:

- Start with 0 vertex. For x the first variable queried, make vertices x, $\overline{x}$ with edges to 0.
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- In the (finite) DAG G constructed, all sources are (weakenings of) axioms of F, and by induction G describes a resolution derivation with weakening.
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Winning Strategy for Spoiler Yields Narrow Proof

Given strategy for Spoiler, build DAG G_π as follows:

- Start with 0 vertex. For x the first variable queried, make vertices x, $\overline{x}$ with edges to 0.
- Inductively, let ρ_v be the unique minimal partial truth value assignment falsifying the clause D_v at v.
- If move on ρ_v is deletion of y, make new vertex $D_v \setminus \{y, \overline{y}\}$ with edge to D_v. Otherwise, if y is queried, make new vertices $D \vee y$, $D \vee \overline{y}$ with edges to D.
- In the (finite) DAG G constructed, all sources are (weakenings of) axioms of F, and by induction G describes a resolution derivation with weakening.
- If we eliminate the weakening we get a derivation in width at most p, since if $|\rho_v| = p + 1$ the next move for Spoiler must be a deletion.

Spoiler Strategy for Tight Proofs

The lower bound on space in terms of width follows from the fact that Spoiler can use proofs in small space to construct winning strategies with few pebbles.

Lemma

Let F be an unsatisfiable CNF formula with

- $W(F) = w$ *and*
- $Sp(F \vdash 0) = s$.

Then

- *Spoiler wins the existential* $(s+w-2)$*-pebble game on* F.

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Proof of Lemma (1 / 2)

Given: proof $\pi = \{\mathbb{C}_0 = \emptyset, \mathbb{C}_1, \ldots, \mathbb{C}_\tau = \{0\}\}$ in space s

Spoiler constructs a strategy by inductively defining partial truth value assignments ρ_t such that ρ_t satisfies $\mathbb{C}_t$ by setting (at most) one literal per clause to true.

W.l.o.g. axiom downloads occur only for $\mathbb{C}_t$ of size $|\mathbb{C}_t| \leq s - 2$.

One memory slot must be saved for the resolvent, otherwise the next step will be an erasure and we can inverse the order of these two derivation steps.

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Proof of Lemma (2 / 2)

- At download of $C \in F$, Spoiler queries Duplicator about all variables in C and keep the literal satisfying it, using at most $(s-2)+w$ pebbles.
- When a clause is deleted, Spoiler deletes the corresponding literal satisfying the clause from ρ_t if necessary (i.e., if $|\rho_t| = |\mathbb{C}_t|$).
- For inference steps, Spoiler sets $\rho_t = \rho_{t-1}$ since by induction ρ_{t-1} must satisfy the resolvent.

Now ρ_τ cannot satisfy $\mathbb{C}_\tau = \{0\}$, so Duplicator must fail at some time prior to τ.

Thus Spoiler has a winning strategy with $\leq (s-2)+w$ pebbles.

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Lower Bound on Space in Terms of Width

Theorem (Atserias & Dalmau 2003)

For any unsatisfiable k-CNF formula F (k fixed) it holds that

$$Sp(F \vdash 0) - 3 \geq W(F \vdash 0) - W(F).$$

Proof.

Combine the facts that:

- If Spoiler wins the existential $(p+1)$-pebble game on F, then $W(F \vdash 0) \leq p$.
- If $W(F) = w$ and $Sp(F \vdash 0) = s$, then Spoiler wins the existential $(s+w-2)$-pebble game on F.

It follows that $W(F \vdash 0) \leq Sp(F \vdash 0) + W(F) - 3$. □

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Open Questions

Atserias & Dalmau say that

Extra space $>$ min 3 needed for any resolution refutation	$\geq$	Extra width $>$ min $W(F)$ needed for any (minimally unsatisfiable) formula

Follow-up questions:

1. Do space and width always coincide?
   Or is there a k-CNF formula family $\{F_n\}_{n=1}^{\infty}$ (for k fixed) such that $Sp(F_n \vdash 0) = \omega(W(F_n \vdash 0))$?
2. Can short resolution proofs be arbitrarily complex w.r.t. space? Or is there a Ben-Sasson-Wigderson-style upper bound on space in terms of length?

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Some References

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Thank you for your attention!