Understanding Conflict-Driven SAT Solving Through the Lens of Proof Complexity

Jakob Nordström

KTH Royal Institute of Technology
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The Satisfiability Problem (SAT)

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Can we use computers to solve the SAT problem efficiently?

SAT Solving in Theory and Practice

The unreasonable effectiveness of SAT solvers

- The Boolean satisfiability problem (SAT) is NP-complete and so should be exponentially hard
- Yet current state-of-the-art conflict-driven clause learning (CDCL) SAT solvers can deal with formulas containing millions of variables
- How can they work so well? What are their limits?

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- Community structure
- Parameterized complexity
- This talk: **proof complexity**
  Rigorous analysis of underlying method of reasoning

Purpose of This Presentation

- Survey some of the research in the area (most of it **not** mine) including some ongoing work (of mine)
- Show some theoretical "benchmark formulas" used to understand potential and limitations of SAT solvers
- Discuss some (of the many) remaining open problems

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Caveats:

- By necessity, selective and somewhat subjective coverage
- Won't do too much name-dropping — full references at end of slides

Some More Caveats and Clarifications

Only basic propositional logic proof search

- No SMT or first-order logic or anything in this talk
- No discussion of preprocessing techniques

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- Asking for rigorous analysis is asking a lot...
- In addition, proof complexity considers optimal algorithms (so restrict focus to unsatisfiable formulas)
- Still possible to prove some highly nontrivial theorems
- Separate question how to interpret these theoretical theorems

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Why theory benchmarks?

- See what SAT solvers can do (sometimes very neat things)
- See what SAT solvers cannot do (provably hard instances)
- See what SAT solvers "should be able" to do (formulas easy for proof system but hard for corresponding SAT solvers)

Outline

1. Resolution and Conflict-Driven Clause Learning
   - The Resolution Proof System
   - Conflict-Driven Clause Learning
   - Theoretical Analysis of CDCL
2. Cutting Planes and Pseudo-Boolean SAT Solving
   - The Cutting Planes Proof System
   - Pseudo-Boolean SAT Solving
3. Seeking Practical CDCL Insights from Theoretical Benchmarks
   - Experimental Set-up
   - Some Tentative Findings

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$ (or $\neg x$)
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (where k is some constant)
- N denotes size of formula (# literals counted with repetitions)
- $\mathcal{O}(f(N))$ grows at most as quickly as $f(N)$ asymptotically
  $\Omega(g(N))$ grows at least as quickly as $g(N)$ asymptotically
  $\Theta(h(N))$ grows equally quickly as $h(N)$ asymptotically

The Resolution Proof System Underlying CDCL

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Done when empty clause $\perp$ derived

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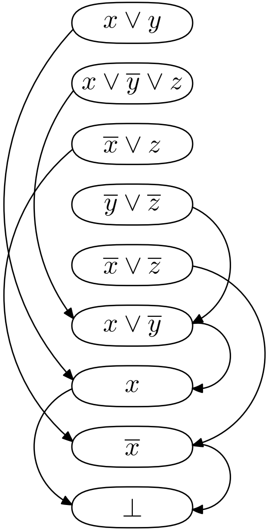

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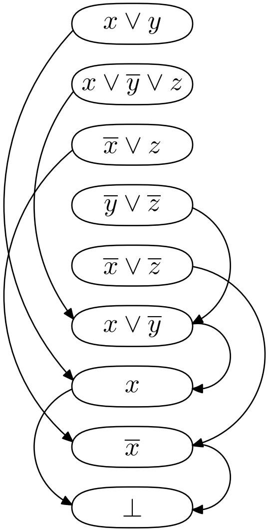
Tree-like resolution if DAG is tree



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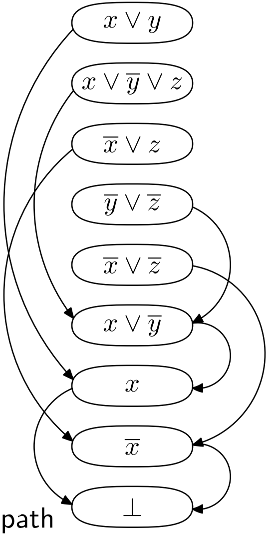
Regular if resolved variables don't repeat on path



Making the Connection to DPLL

Basis of best modern SAT solvers still DPLL method [DP60, DLL62]

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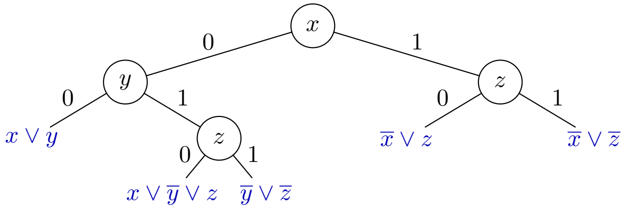
Visualize execution of DPLL algorithm as search tree

- Branch on variable assignments in internal nodes
- Stop in leaves when falsfied clause found



DPLL Execution as Resolution Proof

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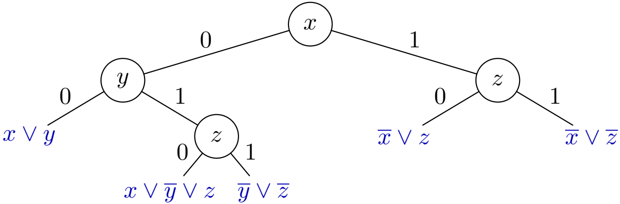
Look at our example again:



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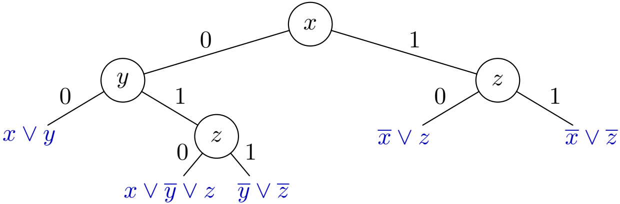

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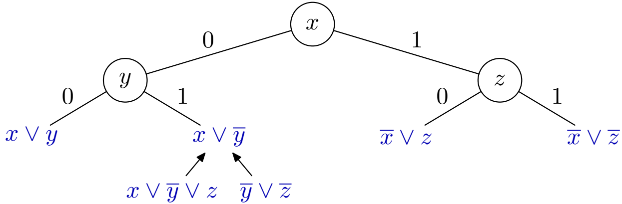

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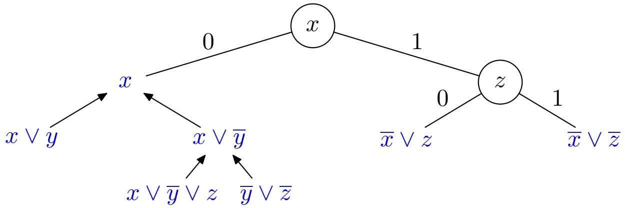

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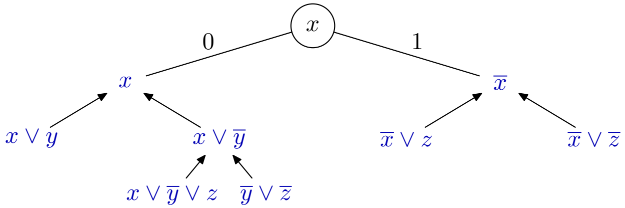

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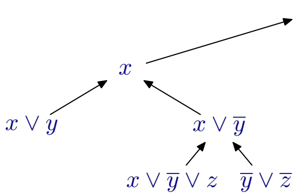

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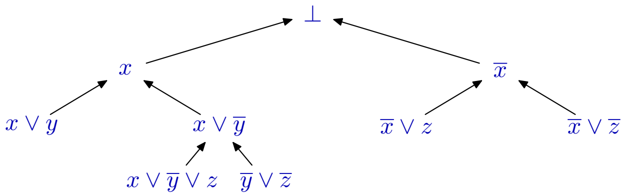

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(Slightly more needed to turn this into formal theorem, but this is essentially it)

CDCL Execution as Resolution Proof

Many more ingredients in modern CDCL SAT solvers [BS97, MS99, MMZ$^+$01], for instance:

- Choice of branching variables crucial
- In leaf, compute & add reason for failure (clause learning)
- Restart every once in a while (saving learned clauses)

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But CDCL still yields resolution proofs
(though clause learning $\Rightarrow$ general DAGs instead of trees)

Will talk more about this later in the presentation

Resolution Size/Length

Size/length of proof = # clauses (9 in our example)
Length of refuting F = min over all proofs for F

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Most fundamental measure in proof complexity

Lower bound on CDCL running time
(can extract resolution proof from execution trace)

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known

Some Examples of Hard Formulas w.r.t. Length (1/3)

Pigeonhole principle (PHP) [Hak85]
"$n+1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
$p_{1,j} \lor p_{2,j} \lor \cdots \lor p_{n+1,j}$	every hole j gets a pigeon

Some Examples of Hard Formulas w.r.t. Length (1/3)

Pigeonhole principle (PHP) [Hak85]
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Even onto functional PHP formula is hard for resolution
"Resolution cannot count"

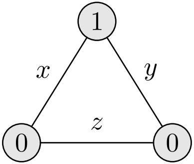
But only length lower bound $\exp(\Omega(\sqrt[3]{N}))$ in terms of formula size

Some Examples of Hard Formulas w.r.t. Length (2/3)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



$$
\begin{aligned}
&(x \lor y) && \land\ (\overline{x} \lor z) \\
\land\ &(\overline{x} \lor \overline{y}) && \land\ (y \lor \overline{z}) \\
\land\ &(x \lor \overline{z}) && \land\ (\overline{y} \lor z)
\end{aligned}
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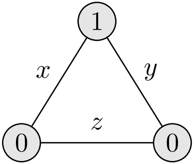

$$
\begin{array}{ll}
(x \lor y) & \land\ (\overline{x} \lor z) \\
\land\ (\overline{x} \lor \overline{y}) & \land\ (y \lor \overline{z}) \\
\land\ (x \lor \overline{z}) & \land\ (\overline{y} \lor z)
\end{array}
$$

Requires length $\exp(\Omega(N))$ on well-connected so-called expanders
"Resolution cannot count $\bmod 2$"

Some Examples of Hard Formulas w.r.t. Length (3/3)

Subset cardinality formulas [Spe10, VS10, MN14]

Variables = 1s in matrix with four 1s per row/column + extra 1
Row ⇒ majority of variables true; column ⇒ majority false

$$\begin{pmatrix}
\mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\
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\mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{1}
\end{pmatrix}$$

$$\begin{aligned}
& (x_{1,1} \lor x_{1,2} \lor x_{1,4}) \\
\land\ & (x_{1,1} \lor x_{1,2} \lor x_{1,8}) \\
\land\ & (x_{1,1} \lor x_{1,4} \lor x_{1,8}) \\
\land\ & (x_{1,2} \lor x_{1,4} \lor x_{1,8}) \\
& \vdots \\
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\land\ & (\overline{x}_{4,11} \lor \overline{x}_{8,11} \lor \overline{x}_{11,11}) \\
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0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\
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\mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & {\color{blue}\mathbf{1}} & 0 & 0 & \mathbf{1}
\end{pmatrix}$$

$$\begin{aligned}
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0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\
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Lower bound $\exp(\Omega(N))$ on expanding matrices (well spread-out)

Resolution Space

Space = max # clauses in memory when performing refutation

Motivated by solver memory usage (but also of intrinsical theory interest)

Can be measured in different ways — makes most sense here to focus on clause space

Space at step t = # clauses at steps $\leq t$ used at steps $\geq t$

1.	$x \vee y$	Axiom
2.	$x \vee \overline{y} \vee z$	Axiom
3.	$\overline{x} \vee z$	Axiom
4.	$\overline{y} \vee \overline{z}$	Axiom
5.	$\overline{x} \vee \overline{z}$	Axiom
6.	$x \vee \overline{y}$	Res$(2, 4)$
7.	x	Res$(1, 6)$
8.	$\overline{x}$	Res$(3, 5)$
9.	$\perp$	Res$(7, 8)$

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Example: Space at step 7 ...

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2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
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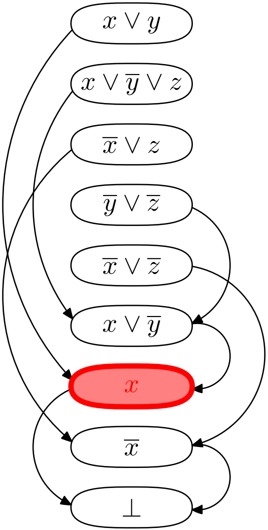

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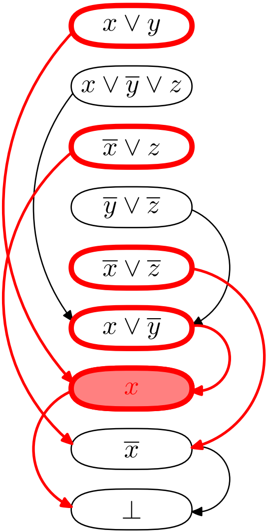

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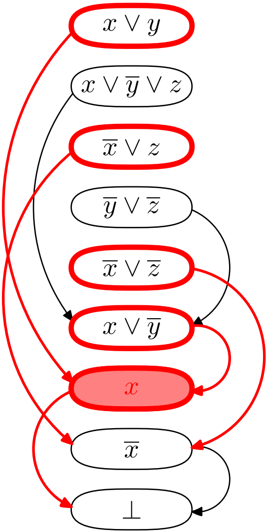

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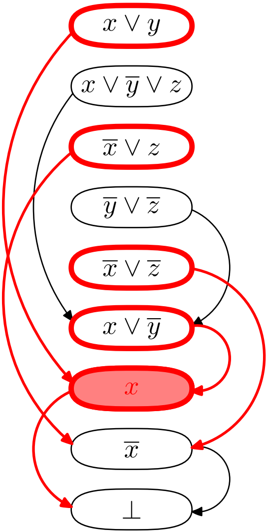
Example: Space at step 7 is 5

Space of proof = max over all steps
Space of refuting F = min over all proofs



Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ (!) [ET01]

Matching $\Omega(N)$ lower bounds known [ABRW02, BG03, ET01]

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Linear space lower bounds might not seem so impressive. . .

But:

- Apply for space on top of storing formula
- Hold even for optimal algorithms that magically know exactly which clauses to throw away or keep
- So significantly more space might be needed in practice
- And linear space upper bound obtained for proofs of exponential size

Length and Space

Exist space-efficient proofs $\Rightarrow$ exist short proofs [AD08]
(for k-CNF formulas, to be precise)

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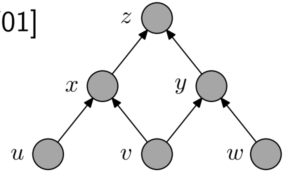
Pebbling formulas [Nor09, NH13, BN08]

- Can be refuted in length $\mathcal{O}(N)$
- May require space $\Omega(N/\log N)$

Pebbling Formulas

Encode so-called pebble games on DAGs [BW01]

1. $u_1 \oplus u_2$
2. $v_1 \oplus v_2$
3. $w_1 \oplus w_2$
4. $(u_1 \oplus u_2) \wedge (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2)$
5. $(v_1 \oplus v_2) \wedge (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2)$
6. $(x_1 \oplus x_2) \wedge (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2)$
7. $\neg(z_1 \oplus z_2)$



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- truth propagates upwards
- but sink is false

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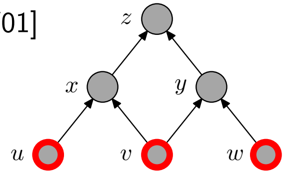

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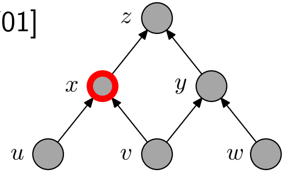

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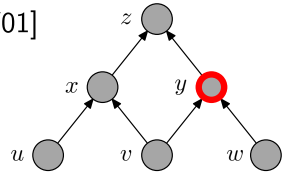

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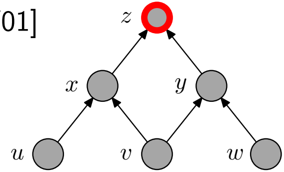

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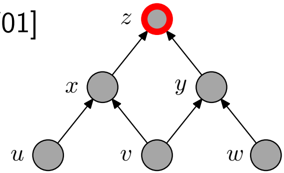

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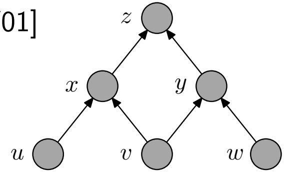

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Write in CNF
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$$(x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \wedge (x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2)$$
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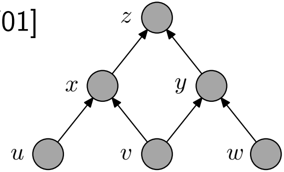

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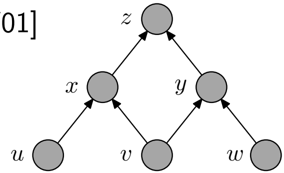

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Pebbling space lower bounds $\Rightarrow$ resolution space lower bounds
(Works also for other functions than $\oplus$)

Length-Space Trade-offs

Length $\approx$ running time; space $\approx$ memory consumption
SAT solvers aggressively try to minimize both — is this possible?

Length-Space Trade-offs

Length ≈ running time; space ≈ memory consumption
SAT solvers aggressively try to minimize both — is this possible?

Theorem ([BN11, BBI12, BNT13])

There are formulas for which

- *exist refutations in short length*
- *exist refutations in small space*
- *optimization of one measure causes dramatic blow-up for other measure*

Holds for

- Pebbling formulas on the right graphs
- Tseitin formulas on long, narrow rectangular grids

So simultaneous optimization not possible [at least in theory]

Abstract Description of CDCL (1/2)

Trail: a stack of decisions $x_i \stackrel{\mathsf{d}}{=} b$ and unit propagations $x_i \stackrel{C}{=} b$

$$\Big(\underbrace{x_7 \stackrel{\mathsf{d}}{=} 0}_{\text{dec. level 1}}, \underbrace{x_2 \stackrel{\mathsf{d}}{=} 1, x_{12} \stackrel{C_1}{=} 0}_{\text{decision level 2}}, \underbrace{x_6 \stackrel{\mathsf{d}}{=} 1, x_4 \stackrel{C_2}{=} 1, x_1 \stackrel{C_3}{=} 0}_{\text{decision level 3}}, \underbrace{x_{11} \stackrel{\mathsf{d}}{=} 0, x_{59} \stackrel{C_4}{=} 1}_{\text{decision level 4}}\Big)$$

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Clause database $\mathcal{D}$: original formula + learned clauses

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else if all variables assigned, output `SAT`;
else if some $C \in \mathcal{D}$ unit w.r.t. trail, move to **Unit**;
else if restart, set trail to $()$ and move to **Default**;
else

1. decide if to apply database reduction to $\mathcal{D}$;
2. move to **Decision**

Abstract Description of CDCL (2/2)

Unit Pick clause $C \in \mathcal{D}$ that is unit w.r.t. trail
Add propagated assignment $x \stackrel{C}{=} b$ to trail
Move to **Default**

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else

- apply learning scheme to derive asserting clause C;
- backjump, i.e., remove decision levels $>$ assertion level of C from trail;
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Move to **Default**

Description from [EJL$^+$16] drawing heavily on [AFT11, BHJ08, PD11]

CDCL Execution Example

Too small formula for interesting example. . .

$$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$$

CDCL Execution Example

Too small formula for interesting example... So expand slightly:

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

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$w \stackrel{\mathsf{d}}{=} 0$

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$w \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{\overline{u} \lor w}{=} 0$

$x \stackrel{\mathsf{d}}{=} 0$

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$$(u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{u} \vee w) \wedge (\overline{u} \vee \overline{w})$$

$w \overset{\mathsf{d}}{=} 0$

$u \overset{\overline{u} \vee w}{=} 0$

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \vee x \vee y}{=} 1$

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$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

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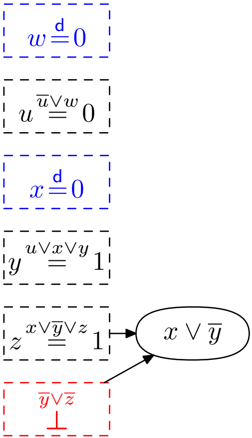

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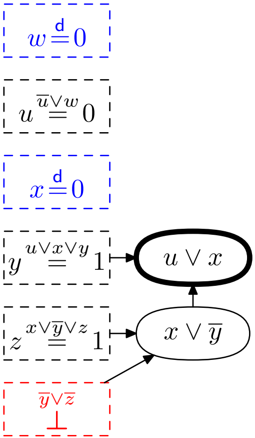

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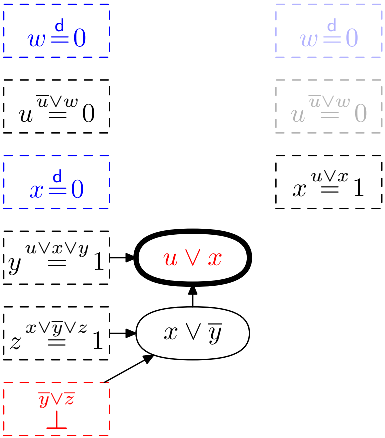

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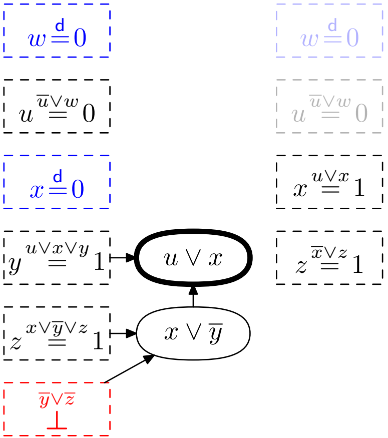

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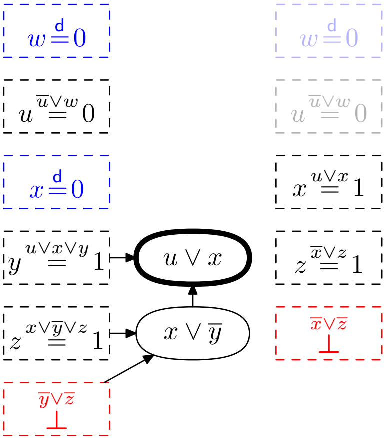

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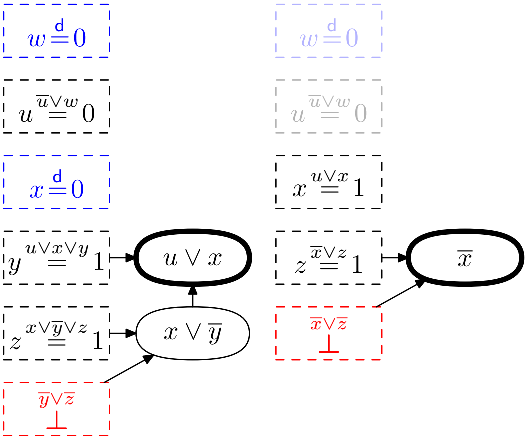

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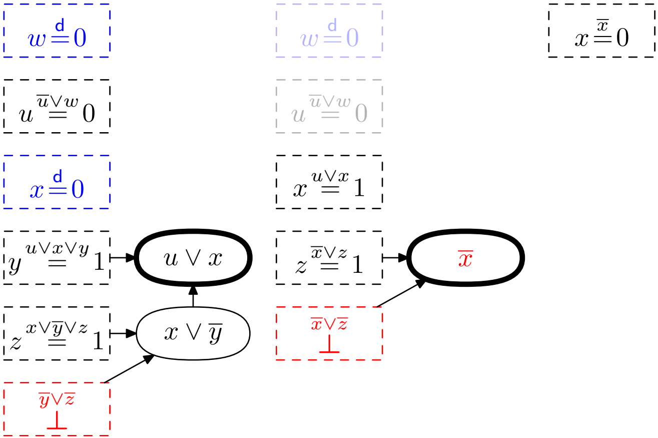

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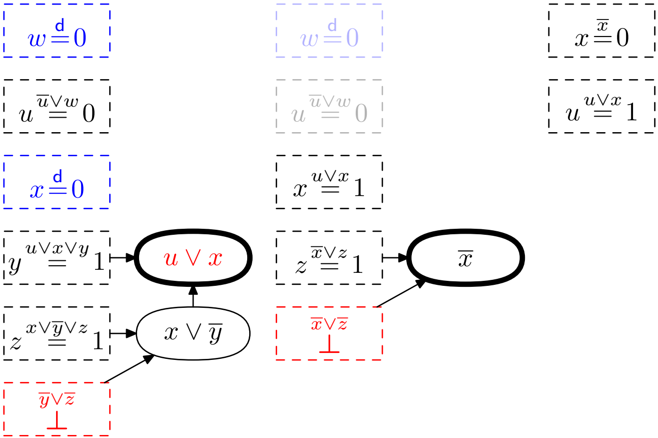

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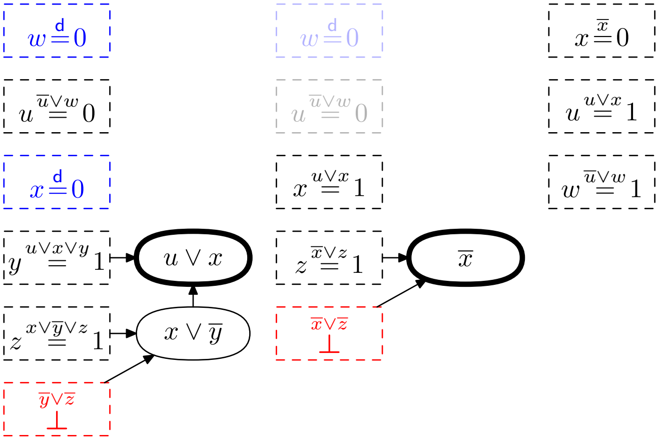

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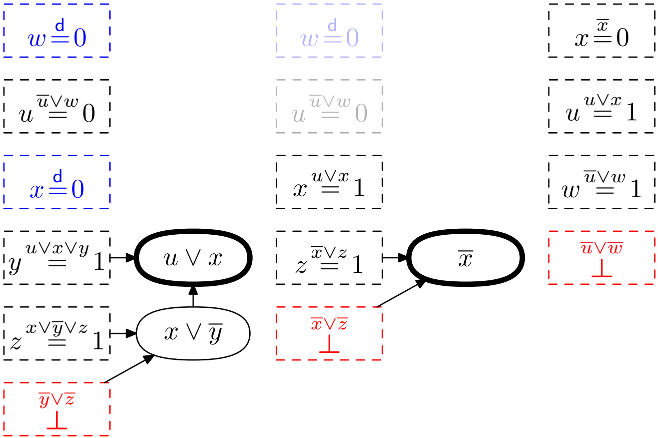

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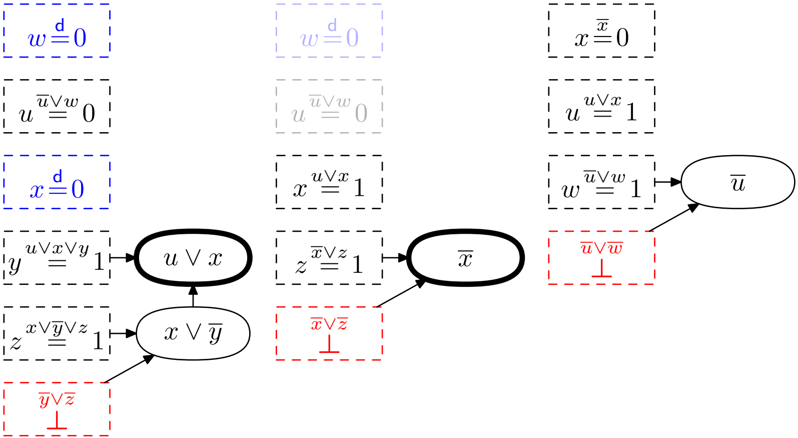

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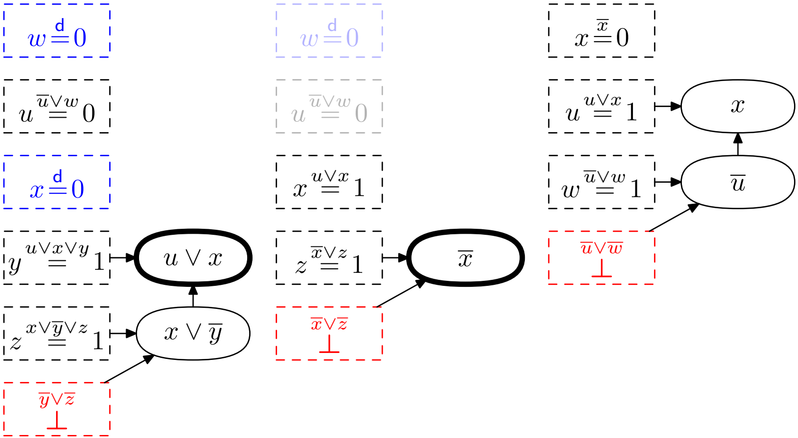

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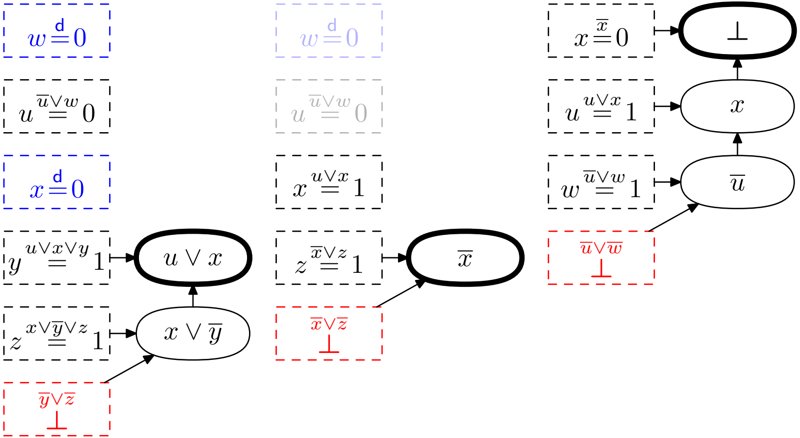

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Obtain resolution refutation...

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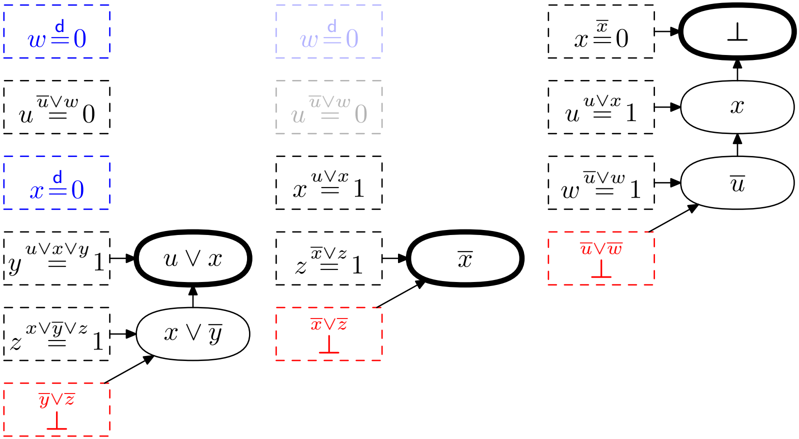

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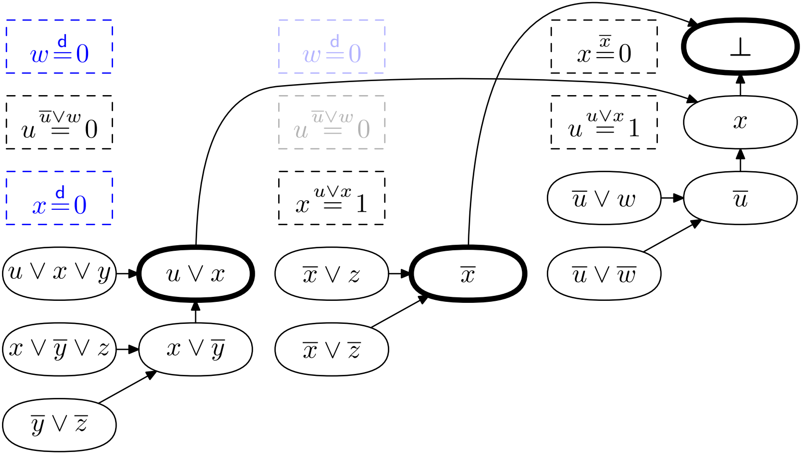

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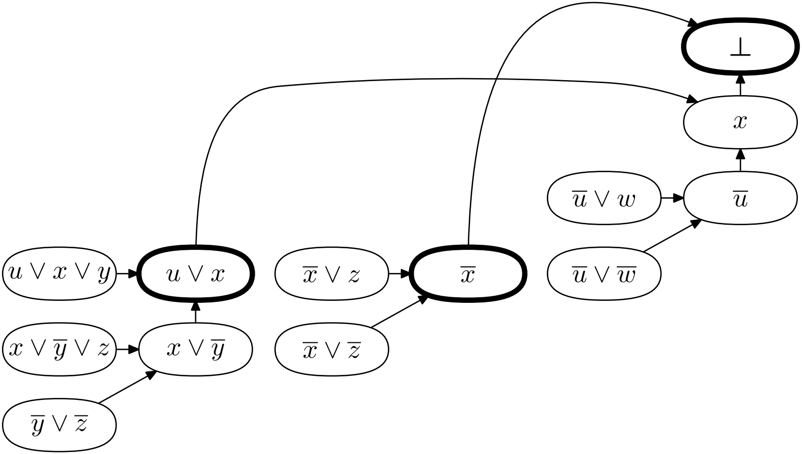

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- Long line of work aimed at proving that CDCL explores resolution search space efficiently, e.g., [BKS04, Van05, BHJ08, HBPV08]
- Challenging problem — progress only by making assumptions such as
  - artificial preprocessing
  - decisions past conflicts
  - non-standard learning scheme
  - no unit propagation(!)

Proof Plan for CDCL Simulation of Resolution

General idea is obvious:

- Given resolution proof $(C_1, C_2, \ldots, C_\tau)$
- Force solver to efficiently learn C_t for $t = 1, 2, 3, \ldots$

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Not as easy as it seems...

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- Suppose have $C \lor x$ and $D \lor \overline{x}$ and now want to learn $C \lor D$
- Why not just decide to make $C \lor D$ false $\Rightarrow$ conflict on x?!
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Non-standard assumptions needed precisely for these reasons

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- Enough to learn other clauses yielding at least same unit propagations as C_t (**absorption**)
- Good, so then we're done understanding CDCL?
  Not quite...

Room for Further Improvement of [AFT11, PD11]? (1/2)

Learning scheme

- Learned clause assertive but otherwise adversarially chosen
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Restart policy

- Restarts are *not too frequent* (unless Luby is too frequent)
- But no progress at all in between restarts
- Restarts seem important, but not quite *that* important?!

Room for Further Improvement of [AFT11, PD11]? (2/2)

Decision strategy

- In [PD11], crucially relies on (unknown) resolution proof
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Simulation efficiency

- Solvers typically have to run in (close to) linear time $\mathcal{O}(n)$
- But simulation will yield something like $\mathcal{O}(n^5)$ running time

What We Would Want

Want a more fine-grained and realistic CDCL model. . .

- Capture restarts, clause learning, memory management, etc.
- Modular design to allow study of different features
- Theoretical analogue of projects in [KSM11, SM11, ENSS16]

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. . . Leading to improved theoretical insights

- Can CDCL proof search be time *and space* efficient?
- And can it be *really* efficient? (No large polynomial blow-ups)
- How does memory management affect proof search quality?
- Do restarts increase reasoning power?
- How do other heuristics help or hinder proof search?

What We Have So Far (1/2)

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- **Time/Size:** # decisions + propagations + conflict analysis steps
  Space: (Size of clause database) − (size of formula)

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- **Cannot locally verify proof** but need global view (doubleplusunnice)

Sanity Check: CDCL Cannot Do Better than Resolution

Theorem ([EJL+16])

If *CDCL with "standard" learning scheme (e.g., 1UIP) decides* F *in time* τ *and space* s
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So lower bounds in resolution trade-offs automatically carry over
But can CDCL find time-efficient and space-efficient proofs?

Time-Space Trade-Offs for CDCL (in Math Notation)

We obtain CDCL analogues of (almost all) trade-off results in [BN11, BBI12, BNT13] — here is one sample:

Theorem ([EJL$^+$16], slightly informal)

For your favourite $k \in \mathbb{N}^+$ $\exists$ explicit formulas F_N of size $\approx N$ such that

- *CDCL with 1UIP learning and no restarts can decide F_N in time $\mathcal{O}(N^k)$ and space $\mathcal{O}(N^k)$*
- *CDCL with 1UIP learning and no restarts can decide F_N in space $\mathcal{O}(\log^2 N)$ and time $N^{\mathcal{O}(\log N)}$*
- *For any CDCL run in time τ and space s using any learning scheme and restart policy it holds that $\tau \gtrapprox (N^k/s)^{\Omega(\log\log N/\log\log\log N)}$*

Time-Space Trade-Offs for CDCL (in English)

Very informal statement of theorem to convey high-level message:

- Somewhat tricky formulas F_N (require superlinear time)
- CDCL can solve them efficiently for generous memory management (even without restarts)
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Interpretation:

- This is only a mathematical theorem about asymptotic properties of theoretical benchmarks
- But have some indications of similar behaviour for scaled-down versions in practical experiments [ENSS16]

Cutting Planes

Introduced in [CCT87] based on integer LP in [Gom63, Chv73]

Clauses interpreted as linear inequalities over the reals with integer coefficients (identifying $1 \equiv \mathit{true}$ and $0 \equiv \mathit{false}$)

Example: $x \vee y \vee \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$

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Derivation rules

Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable

Size, Length and Space

Length = total # lines/inequalities in refutation

Size = sum also size of coefficients

Space = max # lines in memory during refutation

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- is strictly stronger w.r.t. space — can refute any CNF in constant space 5 (!) [GPT15] (But coefficients will be exponentially large — what if also coefficient size counted?)

Hard Formulas w.r.t. Cutting Planes Length

Clique-coclique formulas [Pud97]
"A graph with an m-clique is not $(m{-}1)$-colourable"

$p_{i,j}$ = indicator variables for edges in an n-vertex graph
$q_{k,i}$ = identifiers for members of m-clique in graph
$r_{i,\ell}$ = encoding of legal $(m{-}1)$-colouring of vertices

$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	some vertex is kth member of clique
$\overline{q}_{k,i} \lor \overline{q}_{k',i}$	clique members are uniquely defined
$p_{i,j} \lor \overline{q}_{k,i} \lor \overline{q}_{k',j}$	clique members are connected by edges
$r_{i,1} \lor r_{i,2} \lor \cdots \lor r_{i,m-1}$	every vertex i has a colour
$\overline{p}_{i,j} \lor \overline{r}_{i,\ell} \lor \overline{r}_{j,\ell}$	neighbours have distinct colours

Exponential lower bound via interpolation and circuit complexity
Technique very specifically tied to structure of formula

Open Problems for Cutting Planes Length and Space

Open Problems

Prove length lower bounds for cutting planes

- *for Tseitin formulas*
- *for random k-CNFs*
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Open Problems

Prove space lower bounds for cutting planes

- *with constant-size coefficients (very weak bounds in [GPT15])*
- *with polynomial-size coefficients (nothing known)*

Size-Space Trade-offs for Cutting Planes?

- Short cutting planes refutations of (lifted) Tseitin formulas on expanders need large space [GP14] (but probably don't exist)
- Short cutting planes refutations of (some) pebbling formulas need large space [HN12, GP14] (and such refutations exist)

Results obtained via communication complexity

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Open Problem

Are there trade-offs where the space-efficient CP refutations have small coefficients? (Say, of polynomial or even constant size)

Size-Space Trade-offs for Cutting Planes!

Recent news: Yes, there are such trade-offs!

Theorem ([dRNV16])

There exist flavours of pebbling formulas such that

- $\exists$ *small-size refutations with constant-size coefficients*
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- *Decreasing the space even for refutations with exponentially large coefficients causes exponential blow-up of length*

- Again uses communication complexity (+ several other twists)
- Downside: Parameters worse than in previous results

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Several challenges:

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- Linear constraints more complicated than clauses — and integer arithmetic can become expensive
- Not obvious how to do conflict analysis
  - Can sometimes skip "resolution steps" in conflict analysis with propagating constraints on reason side — good or bad?
  - Can happen that "resolvent" is not conflicting — can be fixed in several ways, but what way is best?

Conflict-Driven CP Solvers: Two Concrete Obstacles

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- **Roadblock 2(?):** Solvers seem inefficient for systems of inequalities that have rational but not integral solutions (too limited form of division?)
- Fail on, e.g., even colouring formulas [Mar06] for no obvious good reason
- Not well understood at all — work in progress

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- Study effect of different CDCL heuristics on performance

Theoretically Easy Combinatorial Benchmarks

- Study tweaked versions of well-studied formulas with:
  - short resolution proofs that can in principle be found by CDCL
  - without any preprocessing
  - often even without any restarts
  - sometimes even without learning, i.e., just DPLL (though might incur some blow-up)
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- Practical note: many (though not quite all) formulas generated using the tool **CNFgen** [CNF, LENV16]

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To run experiments, add “knobs” to Glucose [AS09, Glu] and vary settings for:

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Yields huge number of potential combinations

- Not all combinations make sense, but many do
- Test also settings where "convential wisdom" knows answer

Some Preliminary Conclusions (1/2)

Importance of restarts

- Sometimes very frequent restarts very important
- Crucial in [AFT11, PD11] for CDCL to simulate resolution efficiently
- Also seems to matter in practice for some formulas which are hard for subsystems of resolution such as regular resolution (stone formulas [AJPU07])

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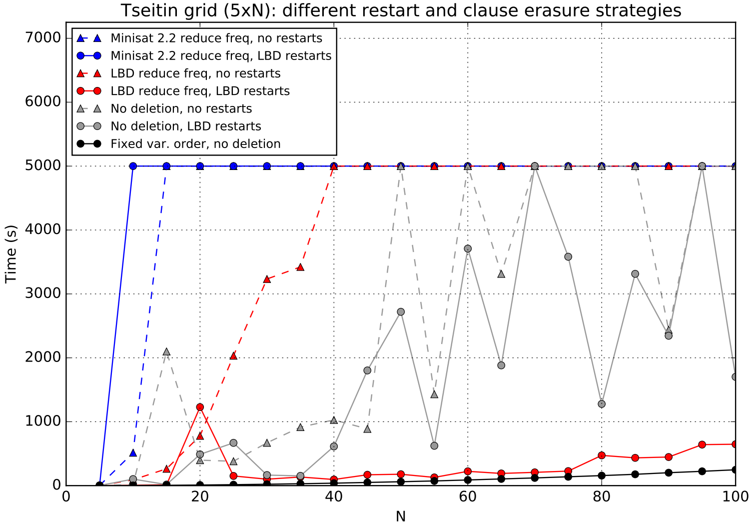
Clause erasure

- Theory says very aggressive clause removal could hurt badly
- Seem to see this on scaled-down versions of time-space trade-off formulas in [BBI12, BNT13] (Tseitin formulas)
- Even no erasure at all can be competitive for these formulas for frequent enough restarts

Plot 1: Tseitin Formulas on Grids



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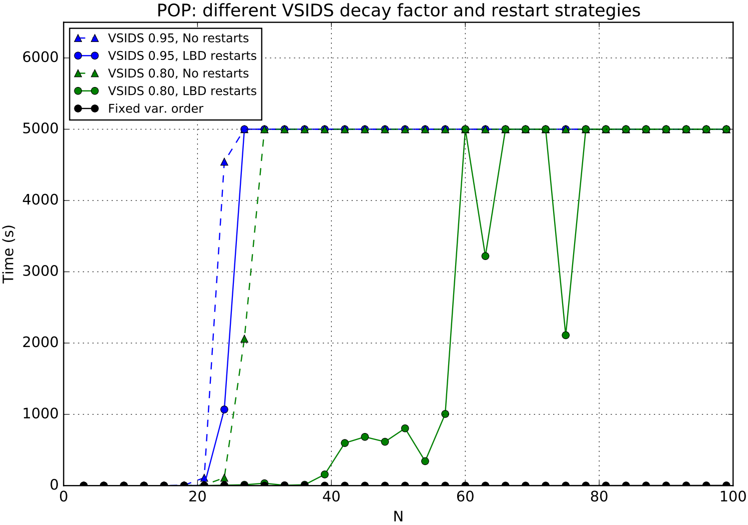
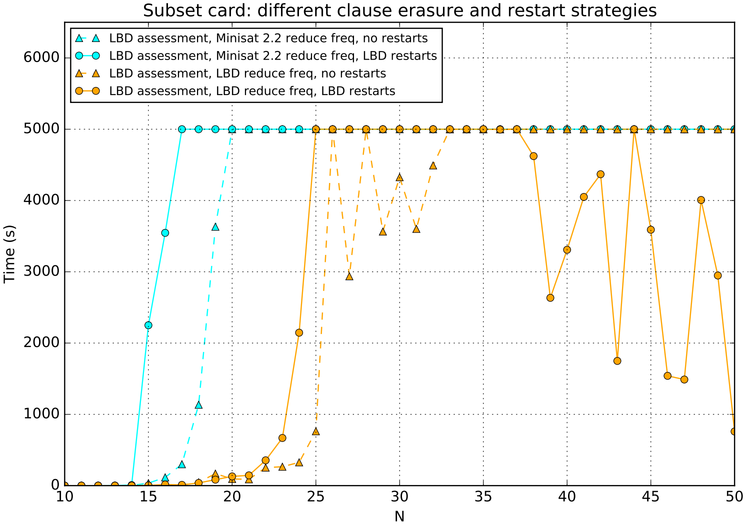
CDCL vs. resolution

- Sometimes CDCL fails miserably on easy formulas (Tseitin, even colouring) — VSIDS just goes dead wrong
- Sometimes strange easy-hard-easy patterns (subset cardinality)

Plot 2: Ordering Principle Formulas



Plot 3: Subset Cardinality Formulas



Summing up

This presentation:

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- Brief discussion of cutting planes and pseudo-Boolean solving
- See survey paper [Nor15] for more details

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- Is standard CDCL without restarts weaker than resolution?
- Are there formulas for which VSIDS goes *provably* wrong?
- Can study of subsystems of cutting planes explain power and limitations of pseudo-Boolean solvers?
- Is it possible to build SAT solvers based on stronger proof systems than resolution that beat CDCL solvers?

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Thank you for your attention!

References I

[ABRW02] Michael Alekhnovich, Eli Ben-Sasson, Alexander A. Razborov, and Avi Wigderson. Space complexity in propositional calculus. *SIAM Journal on Computing*, 31(4):1184–1211, 2002. Preliminary version in *STOC '00*.

[AD08] Albert Atserias and Víctor Dalmau. A combinatorial characterization of resolution width. *Journal of Computer and System Sciences*, 74(3):323–334, May 2008. Preliminary version in *CCC '03*.

[AFT11] Albert Atserias, Johannes Klaus Fichte, and Marc Thurley. Clause-learning algorithms with many restarts and bounded-width resolution. *Journal of Artificial Intelligence Research*, 40:353–373, January 2011. Preliminary version in *SAT '09*.

[AJPU07] Michael Alekhnovich, Jan Johannsen, Toniann Pitassi, and Alasdair Urquhart. An exponential separation between regular and general resolution. *Theory of Computing*, 3(5):81–102, May 2007. Preliminary version in *STOC '02*.

[AR08] Michael Alekhnovich and Alexander A. Razborov. Resolution is not automatizable unless W[P] is tractable. *SIAM Journal on Computing*, 38(4):1347–1363, October 2008. Preliminary version in *FOCS '01*.

References II

[AS09] Gilles Audemard and Laurent Simon. Predicting learnt clauses quality in modern SAT solvers. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI '09)*, pages 399–404, July 2009.

[BBI12] Paul Beame, Chris Beck, and Russell Impagliazzo. Time-space tradeoffs in resolution: Superpolynomial lower bounds for superlinear space. In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC '12)*, pages 213–232, May 2012.

[BG03] Eli Ben-Sasson and Nicola Galesi. Space complexity of random formulae in resolution. *Random Structures and Algorithms*, 23(1):92–109, August 2003. Preliminary version in *CCC '01*.

[BHJ08] Samuel R. Buss, Jan Hoffmann, and Jan Johannsen. Resolution trees with lemmas: Resolution refinements that characterize DLL-algorithms with clause learning. *Logical Methods in Computer Science*, 4(4:13), December 2008.

References III

[BKS04] Paul Beame, Henry Kautz, and Ashish Sabharwal. Towards understanding and harnessing the potential of clause learning. *Journal of Artificial Intelligence Research*, 22:319–351, December 2004. Preliminary version in *IJCAI '03*.

[BLLM14] Armin Biere, Daniel Le Berre, Emmanuel Lonca, and Norbert Manthey. Detecting cardinality constraints in CNF. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 285–301. Springer, July 2014.

[BN08] Eli Ben-Sasson and Jakob Nordström. Short proofs may be spacious: An optimal separation of space and length in resolution. In *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS '08)*, pages 709–718, October 2008.

[BN11] Eli Ben-Sasson and Jakob Nordström. Understanding space in proof complexity: Separations and trade-offs via substitutions. In *Proceedings of the 2nd Symposium on Innovations in Computer Science (ICS '11)*, pages 401–416, January 2011.

References IV

[BNT13] Chris Beck, Jakob Nordström, and Bangsheng Tang. Some trade-off results for polynomial calculus. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing (STOC '13)*, pages 813–822, May 2013.

[BS97] Roberto J. Bayardo Jr. and Robert Schrag. Using CSP look-back techniques to solve real-world SAT instances. In *Proceedings of the 14th National Conference on Artificial Intelligence (AAAI '97)*, pages 203–208, July 1997.

[BW01] Eli Ben-Sasson and Avi Wigderson. Short proofs are narrow—resolution made simple. *Journal of the ACM*, 48(2):149–169, March 2001. Preliminary version in *STOC '99*.

[CCT87] William Cook, Collette Rene Coullard, and György Turán. On the complexity of cutting-plane proofs. *Discrete Applied Mathematics*, 18(1):25–38, November 1987.

[Chv73] Vašek Chvátal. Edmond polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4(1):305–337, 1973.

References V

[CNF] CNFgen formula generator and tools. `https://github.com/MassimoLauria/cnfgen`.

[DLL62] Martin Davis, George Logemann, and Donald Loveland. A machine program for theorem proving. *Communications of the ACM*, 5(7):394–397, July 1962.

[DP60] Martin Davis and Hilary Putnam. A computing procedure for quantification theory. *Journal of the ACM*, 7(3):201–215, 1960.

[dRNV16] Susanna F. de Rezende, Jakob Nordström, and Marc Vinyals. How limited interaction hinders real communication (and what it means for proof and circuit complexity). In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science (FOCS '16)*, pages 295–304, October 2016.

References VI

[EJL+16] Jan Elffers, Jan Johannsen, Massimo Lauria, Thomas Magnard, Jakob Nordström, and Marc Vinyals. Trade-offs between time and memory in a tighter model of CDCL SAT solvers. In *Proceedings of the 19th International Conference on Theory and Applications of Satisfiability Testing (SAT '16)*, volume 9710 of *Lecture Notes in Computer Science*, pages 160–176. Springer, July 2016.

[ENSS16] Jan Elffers, Jakob Nordström, Laurent Simon, and Karem A. Sakallah. Seeking practical CDCL insights from theoretical SAT benchmarks. Manuscript in preparation, 2016.

[ET01] Juan Luis Esteban and Jacobo Torán. Space bounds for resolution. *Information and Computation*, 171(1):84–97, 2001. Preliminary versions of these results appeared in *STACS '99* and *CSL '99*.

[Glu] The Glucose SAT solver. `http://www.labri.fr/perso/lsimon/glucose/`.

[Gom63] Ralph E. Gomory. An algorithm for integer solutions of linear programs. In R.L. Graves and P. Wolfe, editors, *Recent Advances in Mathematical Programming*, pages 269–302. McGraw-Hill, New York, 1963.

References VII

[GP14] Mika Göös and Toniann Pitassi. Communication lower bounds via critical block sensitivity. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing (STOC '14)*, pages 847–856, May 2014.

[GPT15] Nicola Galesi, Pavel Pudlák, and Neil Thapen. The space complexity of cutting planes refutations. In *Proceedings of the 30th Annual Computational Complexity Conference (CCC '15)*, volume 33 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 433–447, June 2015.

[Hak85] Armin Haken. The intractability of resolution. *Theoretical Computer Science*, 39(2-3):297–308, August 1985.

[HBPV08] Philipp Hertel, Fahiem Bacchus, Toniann Pitassi, and Allen Van Gelder. Clause learning can effectively P-simulate general propositional resolution. In *Proceedings of the 23rd National Conference on Artificial Intelligence (AAAI '08)*, pages 283–290, July 2008.

References VIII

[HN12] Trinh Huynh and Jakob Nordström. On the virtue of succinct proofs: Amplifying communication complexity hardness to time-space trade-offs in proof complexity (Extended abstract). In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC '12)*, pages 233–248, May 2012.

[JMNŽ12] Matti Järvisalo, Arie Matsliah, Jakob Nordström, and Stanislav Živný. Relating proof complexity measures and practical hardness of SAT. In *Proceedings of the 18th International Conference on Principles and Practice of Constraint Programming (CP '12)*, volume 7514 of *Lecture Notes in Computer Science*, pages 316–331. Springer, October 2012.

[Kri85] Balakrishnan Krishnamurthy. Short proofs for tricky formulas. *Acta Informatica*, 22(3):253–275, August 1985.

[KSM11] Hadi Katebi, Karem A. Sakallah, and João P. Marques-Silva. Empirical study of the anatomy of modern SAT solvers. In *Proceedings of the 14th International Conference on Theory and Applications of Satisfiability Testing (SAT '11)*, volume 6695 of *Lecture Notes in Computer Science*, pages 343–356. Springer, June 2011.

References IX

[LENV16] Massimo Lauria, Jan Elffers, Jakob Nordström, and Marc Vinyals. CNFgen: a generator of crafted CNF formulas. Manuscript in preparation, 2016.

[LP10] Daniel Le Berre and Anne Parrain. The Sat4j library, release 2.2. *Journal on Satisfiability, Boolean Modeling and Computation*, 7:59–64, 2010.

[Mar06] Klas Markström. Locality and hard SAT-instances. *Journal on Satisfiability, Boolean Modeling and Computation*, 2(1-4):221–227, 2006.

[MMZ+01] Matthew W. Moskewicz, Conor F. Madigan, Ying Zhao, Lintao Zhang, and Sharad Malik. Chaff: Engineering an efficient SAT solver. In *Proceedings of the 38th Design Automation Conference (DAC '01)*, pages 530–535, June 2001.

[MN14] Mladen Mikša and Jakob Nordström. Long proofs of (seemingly) simple formulas. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 121–137. Springer, July 2014.

References X

[MS99] João P. Marques-Silva and Karem A. Sakallah. GRASP: A search algorithm for propositional satisfiability. *IEEE Transactions on Computers*, 48(5):506–521, May 1999. Preliminary version in *ICCAD '96*.

[NH13] Jakob Nordström and Johan Håstad. Towards an optimal separation of space and length in resolution. *Theory of Computing*, 9:471–557, May 2013. Preliminary version in *STOC '08*.

[Nor09] Jakob Nordström. Narrow proofs may be spacious: Separating space and width in resolution. *SIAM Journal on Computing*, 39(1):59–121, May 2009. Preliminary version in *STOC '06*.

[Nor15] Jakob Nordström. On the interplay between proof complexity and SAT solving. *ACM SIGLOG News*, 2(3):19–44, July 2015.

[PD11] Knot Pipatsrisawat and Adnan Darwiche. On the power of clause-learning SAT solvers as resolution engines. *Artificial Intelligence*, 175:512–525, February 2011. Preliminary version in *CP '09*.

References XI

[Pud97] Pavel Pudlák. Lower bounds for resolution and cutting plane proofs and monotone computations. *Journal of Symbolic Logic*, 62(3):981–998, September 1997.

[SM11] Karem A. Sakallah and João Marques-Silva. Anatomy and empirical evaluation of modern SAT solvers. *Bulletin of the European Association for Theoretical Computer Science*, 103:96–121, February 2011.

[Spe10] Ivor Spence. sgen1: A generator of small but difficult satisfiability benchmarks. *Journal of Experimental Algorithmics*, 15:1.2:1–1.2:15, March 2010.

[Stå96] Gunnar Stålmarck. Short resolution proofs for a sequence of tricky formulas. *Acta Informatica*, 33(3):277–280, May 1996.

[Urq87] Alasdair Urquhart. Hard examples for resolution. *Journal of the ACM*, 34(1):209–219, January 1987.

References XII

[Van05] Allen Van Gelder. Pool resolution and its relation to regular resolution and DPLL with clause learning. In *Proceedings of the 12th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR '05)*, volume 3835 of *Lecture Notes in Computer Science*, pages 580–594. Springer, 2005.

[VS10] Allen Van Gelder and Ivor Spence. Zero-one designs produce small hard SAT instances. In *Proceedings of the 13th International Conference on Theory and Applications of Satisfiability Testing (SAT '10)*, volume 6175 of *Lecture Notes in Computer Science*, pages 388–397. Springer, July 2010.