Proof Complexity and SAT Solving

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"Theory and Practice of SAT and Combinatorial Solving"
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The Boolean Satisfiability (SAT) Problem

SAT

Given a propositional logic formula F, is there a satisfying assignment for F?

$$
\begin{aligned}
&(x \lor z) \land (y \lor \lnot z) \land (x \lor \lnot y \lor u) \land (\lnot y \lor \lnot u) \\
\land\ &(u \lor v) \land (\lnot x \lor \lnot v) \land (\lnot u \lor w) \land (\lnot x \lor \lnot u \lor \lnot w)
\end{aligned}
$$

- Variables should be set to **true** or **false**
- Constraint $(x \lor \lnot y \lor z)$: means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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Can we use computers to solve this problem efficiently?

The Same Problem in Three Different Shapes

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$$\begin{aligned}
(1-x)(1-z) &= 0 \\
(1-y)z &= 0 \\
(1-x)y(1-u) &= 0 \\
yu &= 0 \\
(1-u)(1-v) &= 0 \\
xv &= 0 \\
u(1-w) &= 0 \\
xuw &= 0
\end{aligned}$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

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$$(x \vee z) \wedge (y \vee \neg z) \wedge (x \vee \neg y \vee u) \wedge (\neg y \vee \neg u)$$
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$$\begin{aligned}
1 - x - z + xz &= 0 \\
z - yz &= 0 \\
y - xy - yu + xyu &= 0 \\
yu &= 0 \\
1 - u - v + uv &= 0 \\
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u - uw &= 0 \\
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$1 - x - z + xz = 0$	$x + z \geq 1$
$z - yz = 0$	$y + (1 - z) \geq 1$
$y - xy - yu + xyu = 0$	$x + (1 - y) + u \geq 1$
$yu = 0$	$(1 - y) + (1 - u) \geq 1$
$1 - u - v + uv = 0$	$u + v \geq 1$
$xv = 0$	$(1 - x) + (1 - v) \geq 1$
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$$\begin{aligned} x + z &\geq 1 \\ y - z &\geq 0 \\ x - y + u &\geq 0 \\ -y - u &\geq -1 \\ u + v &\geq 1 \\ -x - v &\geq -1 \\ -u + w &\geq 0 \\ -x - u - w &\geq -2 \end{aligned}$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0, 1\}$-valued solution?

Solving SAT in Theory and Practice

- Problem mentioned in Gödel's letter in 1956 to von Neumann
- Topic of intense research in computer science ever since 1960s
- NP-complete, so probably very hard worst case [Coo71, Lev73]
- But enormous progress last 20–25 years on conflict-driven clause learning (CDCL) SAT solvers [BS97, MS99, MMZ$^+$01]
- Today large-scale real-world problems with hundreds of thousands or millions of variables solved routinely
- But... There are also small formulas (just $\sim$100 variables) that are completely beyond reach for even the very best SAT solvers

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How can we rigorously analyse SAT solving algorithms?

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How can we rigorously analyse SAT solving algorithms?

This talk: Use proof complexity (not only conceivable answer)

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For any algorithm deciding satisfiability formula F, describe which rules of reasoning it uses

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Focus of this talk: Question 1 for different proof systems/algorithms
Study unsatisfiable formulas — proof of satisfiability easy

Outline

1. DPLL, CDCL, and Resolution
   - Davis-Putnam-Logemann-Loveland (DPLL) Method
   - Conflict-Driven Clause Learning (CDCL)
   - Resolution Proof System
2. Algebraic and Semi-algebraic Approaches
   - Nullstellensatz
   - Polynomial Calculus and Gröbner Bases
   - Cutting Planes and Pseudo-Boolean Solving
3. Some Proof Systems We Won't Have Time for
   - Sherali-Adams and Sums of Squares
   - Stabbing Planes
   - Extended Resolution

Formal Description of SAT Problem

- Variable x: takes value **true** $(= 1)$ or **false** $(= 0)$
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

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Given a CNF formula F, is it satisfiable?

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For instance, what about our example formula?

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DPLL: Attempting Smart Case Analysis

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DPLL (somewhat simplified description)

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2. If F contains no clauses, report "satisfiable" and terminate
3. Otherwise pick some variable x in F
4. Set $x = 0$, simplify F and make recursive call
5. Set $x = 1$, simplify F and make recursive call
6. If result in both cases "unsatisfiable", then report "unsatisfiable" and return

A DPLL Toy Example

$$
\begin{aligned}
F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\
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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when falsified clause found (i.e., when conflict reached)

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals

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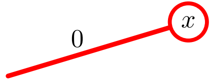

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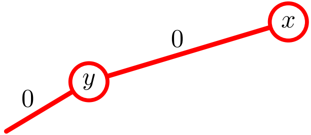

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$$F = (x \lor z) \land (\overline{z}) \land (\overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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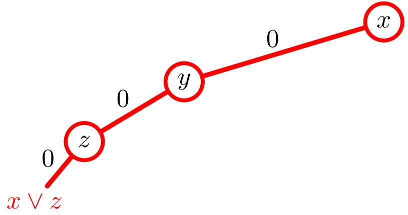

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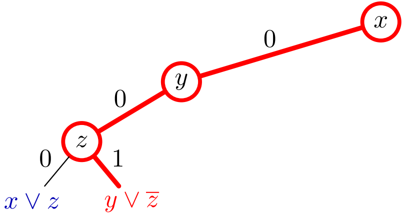

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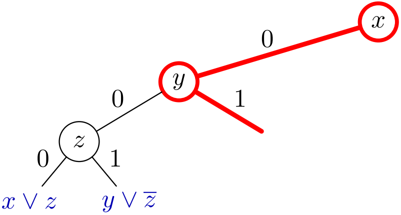

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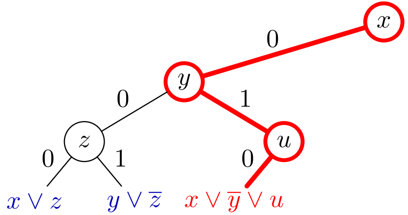

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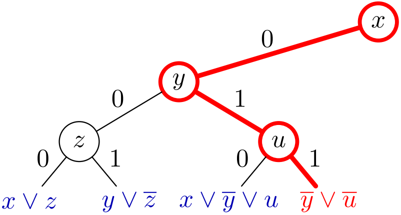

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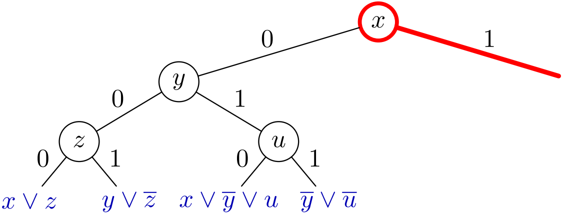

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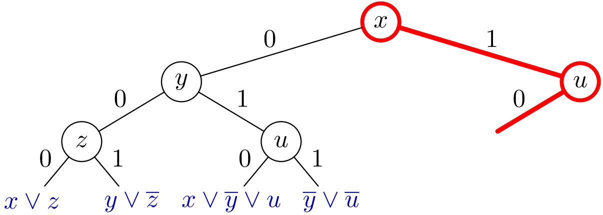

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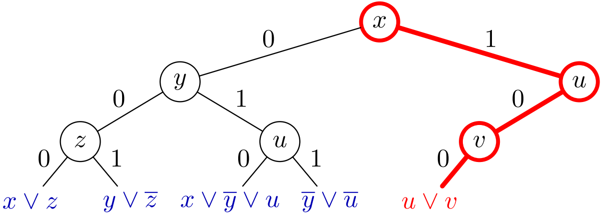

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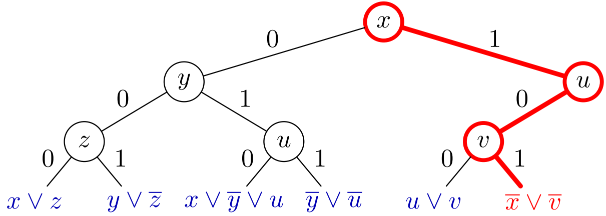

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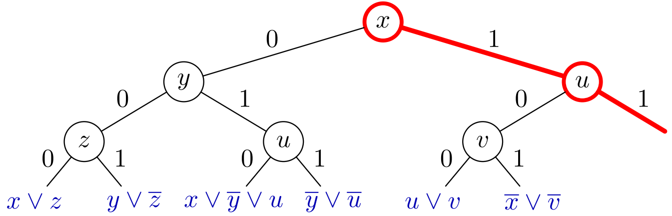

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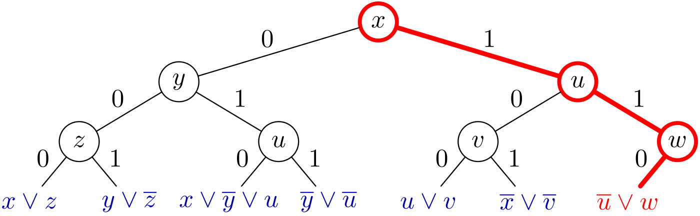

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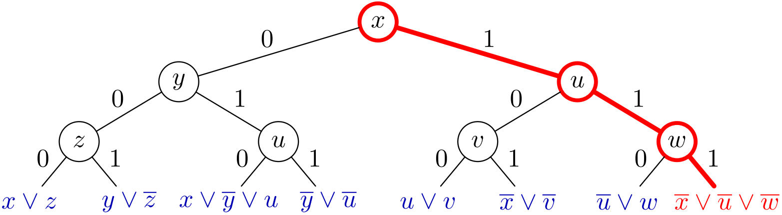


A DPLL Toy Example

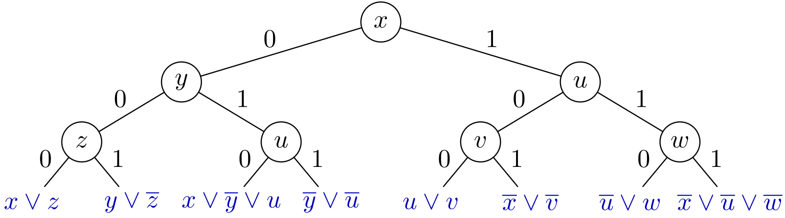
$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when falsified clause found (i.e., when conflict reached)

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



State-of-the-art SAT solvers: Ingredients

Many more ingredients in modern conflict-driven clause learning (CDCL) SAT solvers (as pioneered in [BS97, MS99, MMZ+01]), e.g.:

- Branching or decision heuristic (choice of pivot variables crucial)
- When reaching leaf, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)

Let us briefly discuss some of these ingredients

Variable Assignment Heuristics

Unit propagation

- Suppose current assignment ρ falsifies all literals in $C = \ell_1 \lor \ell_2 \lor \cdots \lor \ell_k$ except one (say ℓ_k) — C is unit under ρ
- Then ℓ_k has to be true, so set it to true
- Known as unit progagation or Boolean constraint progagation
- Always propagate if possible — in modern solvers aim for 99% of assignments being unit propagations

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VSIDS (Variable state independent decaying sum)

- When backtracking, score $+1$ for variables "causing conflict"
- Also multiply all scores with factor $\kappa < 1$ — exponential filter rewarding variables involved in recent conflicts
- When no propagations, decide on variable with highest score

Clause Learning

- At conflict, want to add clause avoiding same part of search tree being explored again

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- In practice, more advanced learning schemes

Clause Learning

- At conflict, want to add clause avoiding same part of search tree being explored again
- Suppose we can compute that decisions $x = 1$, $y = 0$, $z = 1$ responsible for conflict
- Then can add $\overline{x} \vee y \vee \overline{z}$ to avoid these decisions being made again — decision learning scheme
- In practice, more advanced learning schemes
- Derive new clause from clauses unit propagating on the way to conflict

Decisions, Unit Propagations, and Conflict

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

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Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

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$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

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Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

Decisions, Unit Propagations, and Conflict

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$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

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$\boxed{r \stackrel{q \vee r}{=} 1}$

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$u \stackrel{p \lor \overline{u}}{=} 0$

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$w \stackrel{\overline{r} \lor w}{=} 1$

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$y \stackrel{u \lor x \lor y}{=} 1$

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$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

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Notation $p \stackrel{\mathsf{d}}{=} 0$

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$w \stackrel{\overline{r} \vee w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\stackrel{\overline{y} \vee \overline{z}}{\bot}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

Decisions, Unit Propagations, and Conflict

Two kinds of assignments — illustrate on example formula:

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$\boxed{p \overset{\mathsf{d}}{=} 0}$
$u \overset{p \vee \overline{u}}{=} 0$
decision level 1

$\boxed{q \overset{\mathsf{d}}{=} 0}$
$r \overset{q \vee r}{=} 1$
$w \overset{\overline{r} \vee w}{=} 1$
decision level 2

$\boxed{x \overset{\mathsf{d}}{=} 0}$
$y \overset{u \vee x \vee y}{=} 1$
$z \overset{x \vee \overline{y} \vee z}{=} 1$
$\overset{\overline{y} \vee \overline{z}}{\bot}$
decision level 3

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \overset{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\left.\begin{array}{l} \boxed{p \stackrel{\mathsf{d}}{=} 0} \\ u \stackrel{p \vee \overline{u}}{=} 0 \end{array}\right\}$ decision level 1

$\left.\begin{array}{l} \boxed{q \stackrel{\mathsf{d}}{=} 0} \\ r \stackrel{q \vee r}{=} 1 \\ w \stackrel{\overline{r} \vee w}{=} 1 \end{array}\right\}$ decision level 2

$\left.\begin{array}{l} \boxed{x \stackrel{\mathsf{d}}{=} 0} \\ y \stackrel{u \vee x \vee y}{=} 1 \\ z \stackrel{x \vee \overline{y} \vee z}{=} 1 \\ \stackrel{\overline{y} \vee \overline{z}}{\bot} \end{array}\right\}$ decision level 3

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\stackrel{\overline{y} \vee \overline{z}}{\bot}$

decision level 3

Could backtrack by removing last decision level & flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$
$u \stackrel{p \lor \overline{u}}{=} 0$
decision level 1

$q \stackrel{\mathsf{d}}{=} 0$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$
decision level 2

$x \stackrel{\mathsf{d}}{=} 0$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$
decision level 3

Could backtrack by removing last decision level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

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$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

$x \lor \overline{y}$

Could backtrack by removing last decision level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

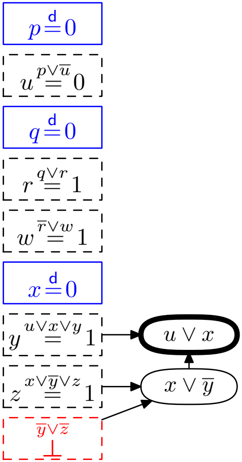
Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \lor \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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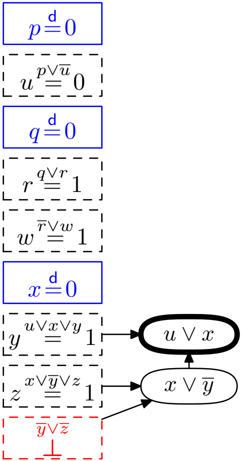
- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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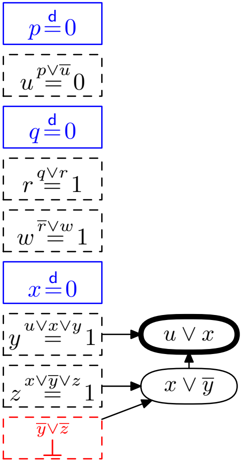


Assertion level 1 (max for non-UIP literal in learned clause) — keep trail to that level

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

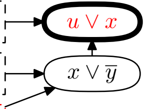
$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$



$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$x \stackrel{u \lor x}{=} 1$

Assertion level 1 (max for non-UIP literal in learned clause) — keep trail to that level

Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

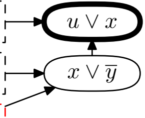
$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$



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Assertion level 1 (max for non-UIP literal in learned clause) — keep trail to that level

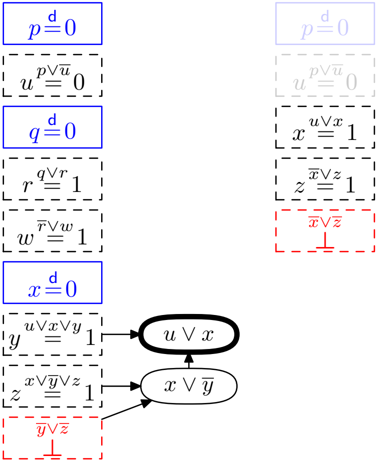
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Then continue as before. . .

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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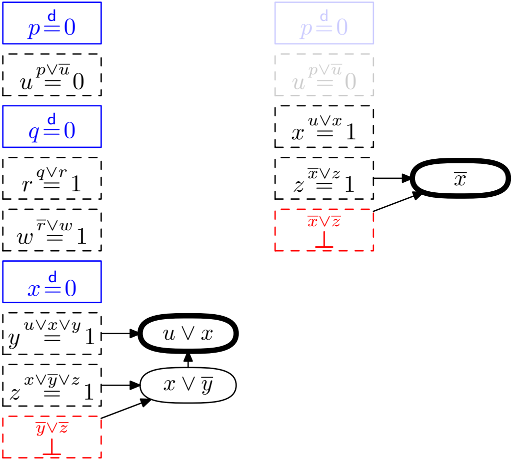

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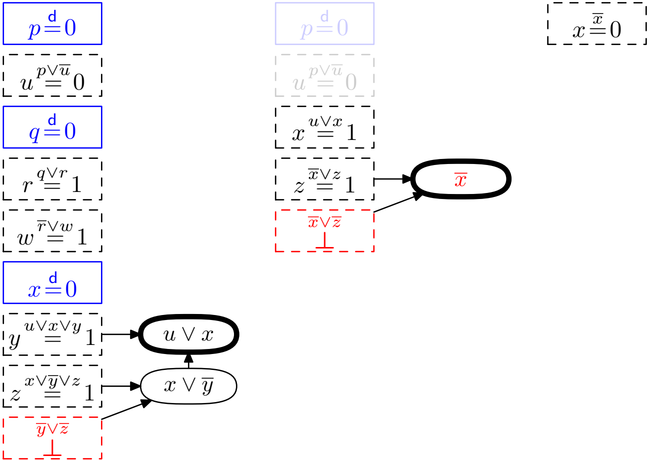

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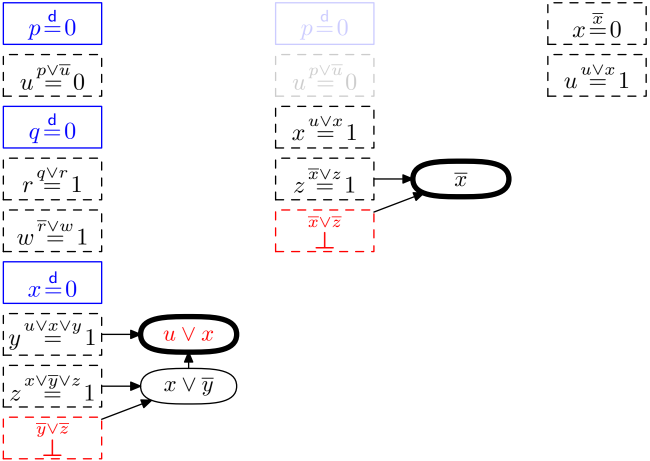

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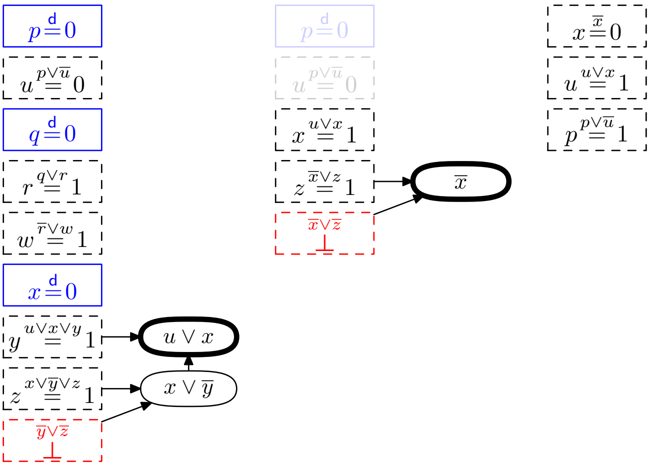

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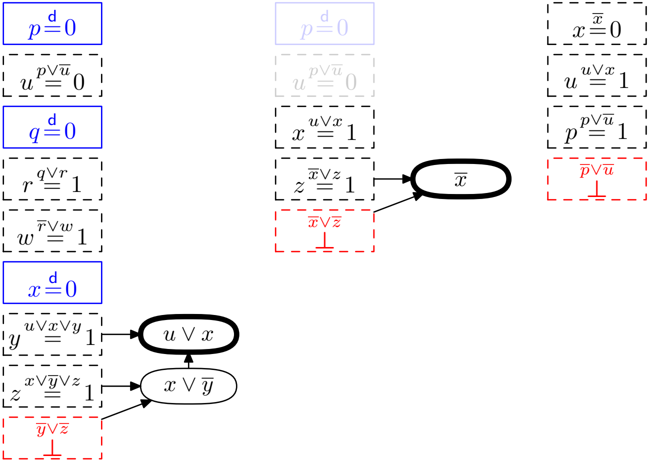

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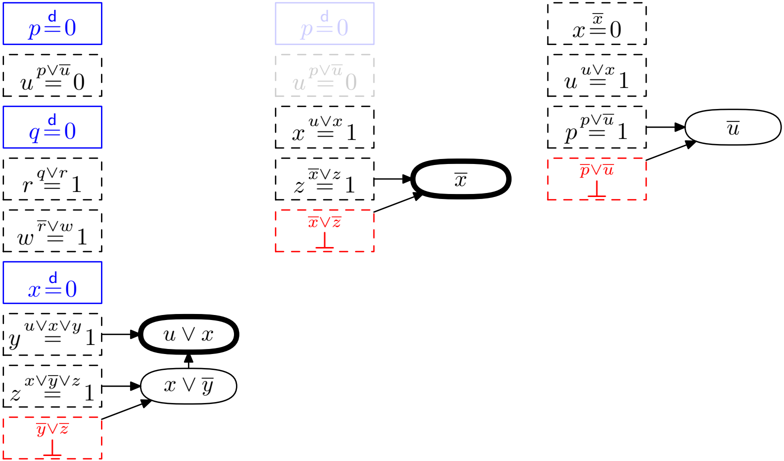

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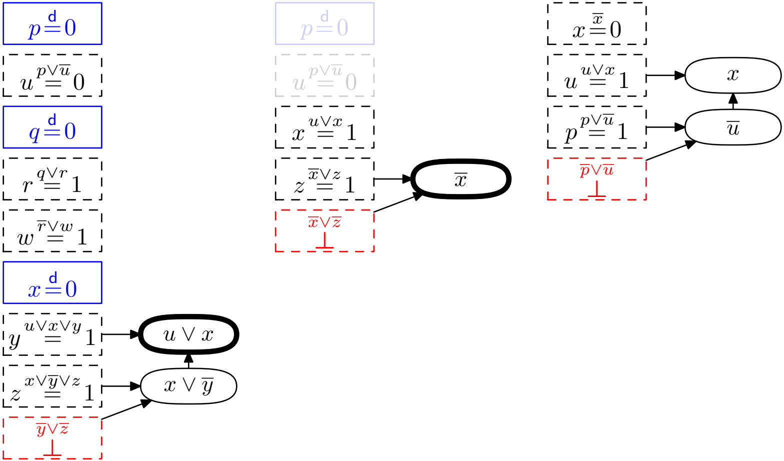

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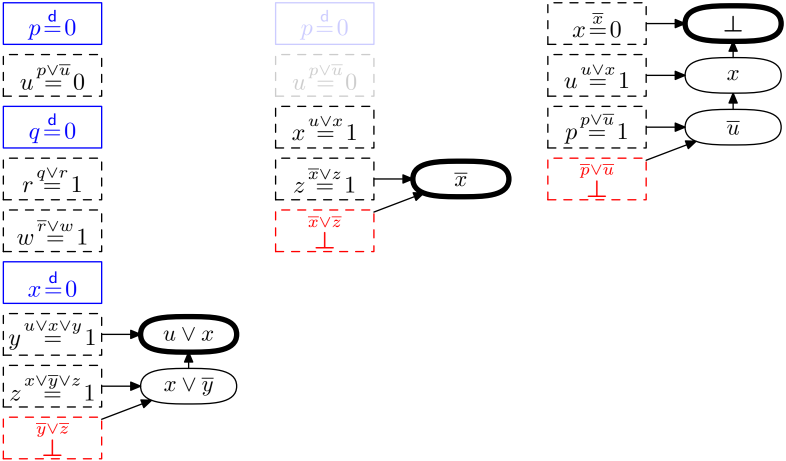

SAT Solver Analysis and the Resolution Proof System

How to make rigorous analysis of SAT solver performance?

Many intricate, hard-to-understand heuristics

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Resolution Proofs by Contradction

Resolution rule:

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Observation

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Such proof by contradiction also called resolution refutation

DPLL and Resolution Proofs

A DPLL execution is essentially a resolution proof

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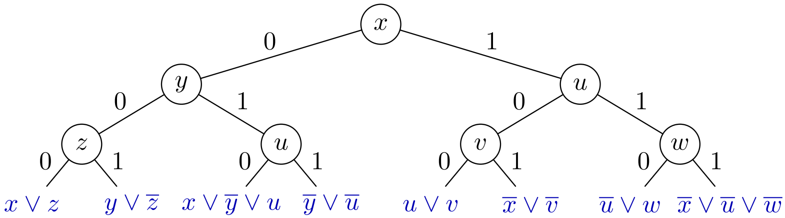

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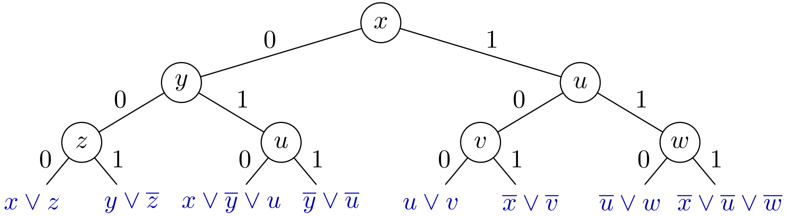

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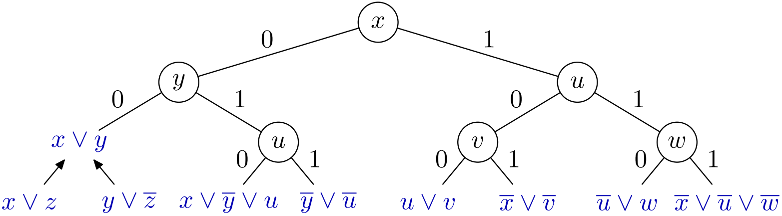

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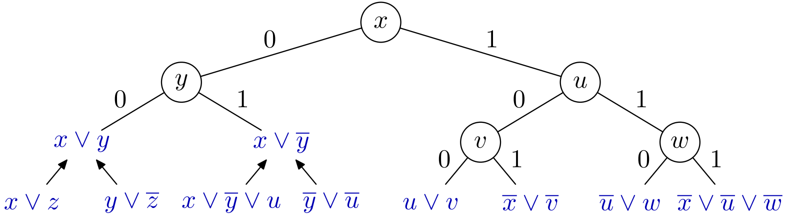

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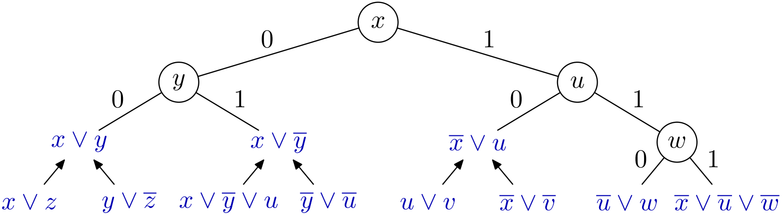

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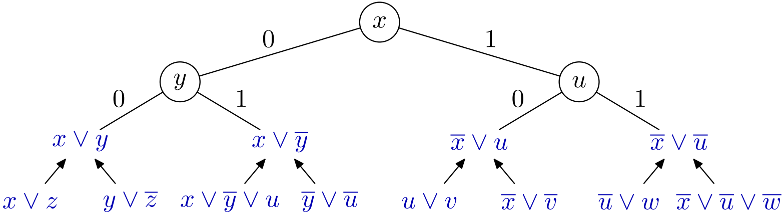

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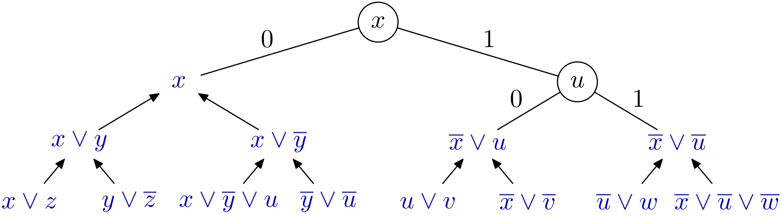

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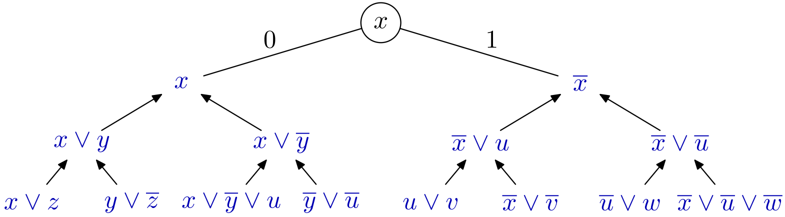

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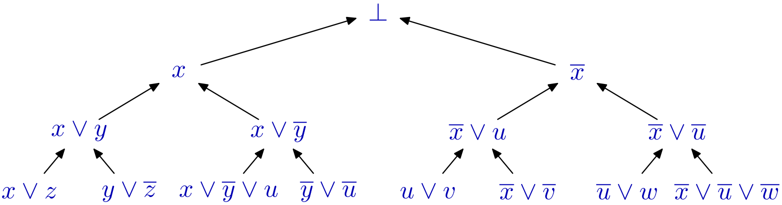

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- Conflict-driven clause learning adds "shortcut edges" in tree, but still yields resolution proof

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Obtain resolution proof. . .

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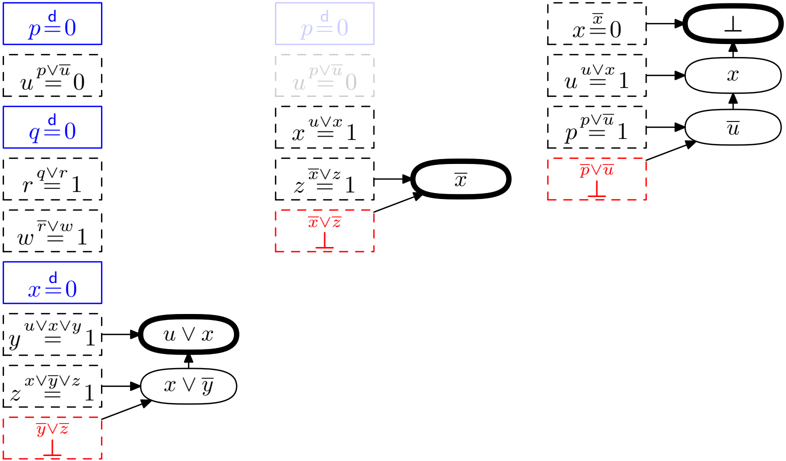
Obtain resolution proof from our example CDCL execution...



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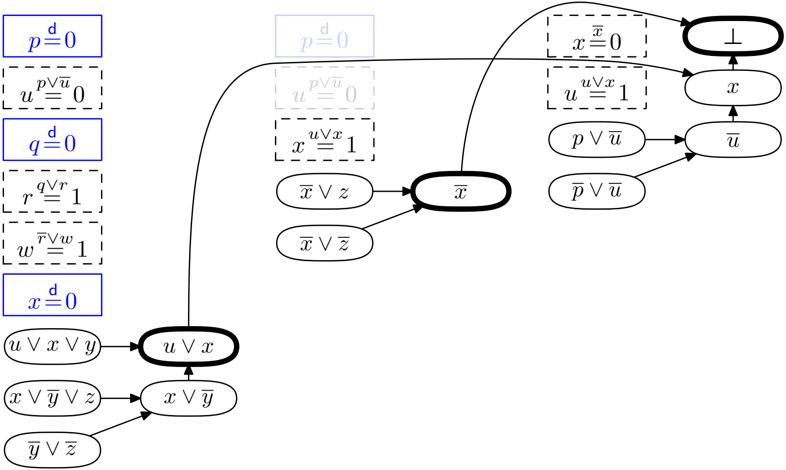

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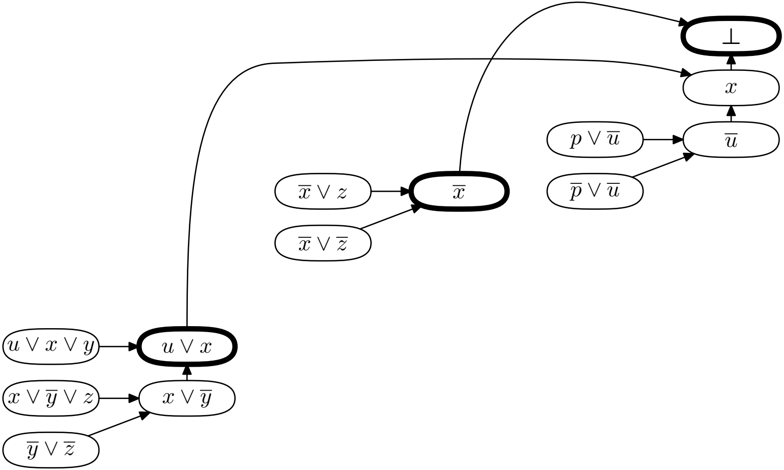

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(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details...

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- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) "obvious" formulas

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Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp\big(\Omega\big(\sqrt[3]{N}\big)\big)$ clauses (measured in terms of formula size N)

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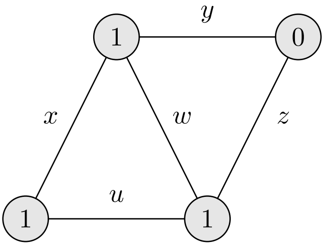

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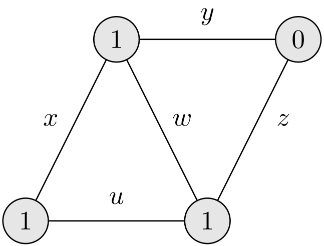

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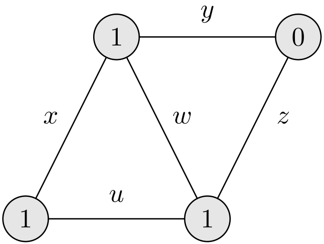

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Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs — **"resolution cannot count** $\mathrm{mod}\ 2$**"**

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And more...

- COLOURING [BCMM05]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera... (See, e.g., [BN21] for overview)

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to polynomial equations

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Hilbert's Nullstellensatz

Consider any system of polynomial equations

$$\begin{aligned} p_1(x_1, \ldots, x_n) &= 0 \qquad & x_1^2 - x_1 &= 0 \\ p_2(x_1, \ldots, x_n) &= 0 \qquad & x_2^2 - x_2 &= 0 \\ &\vdots & &\vdots \\ p_m(x_1, \ldots, x_n) &= 0 \qquad & x_n^2 - x_n &= 0 \end{aligned}$$

in polynomial ring over field $\mathbb{F}$

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Hilbert's Nullstellensatz

System infeasible $\Leftrightarrow$ exist $q_i, r_j \in \mathbb{F}[x_1, \ldots, x_n]$ such that

$$\sum_{i=1}^{m} q_i(x_1, \ldots, x_n) \cdot p_i(x_1, \ldots, x_n) + \sum_{j=1}^{n} r_j(x_1, \ldots, x_n) \cdot (x_j^2 - x_j) = 1$$

Nullstellensatz Proof System [BIK+94]

Nullstellensatz refutation of

$$
\begin{aligned}
p_i(x_1, \ldots, x_n) &= 0 \qquad && i \in [m] \\
x_j^2 - x_j &= 0 \qquad && j \in [n]
\end{aligned}
$$

is (syntactic) equality

$$\sum_{i=1}^{m} q_i(x_1, \ldots, x_n) \cdot p_i(x_1, \ldots, x_n) + \sum_{j=1}^{n} r_j(x_1, \ldots, x_n) \cdot (x_j^2 - x_j) = 1$$

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Complexity measures of refutations:

- Size: number of monomials (when all polynomials expanded out)
- Degree: highest total degree of any polynomial

Nullstellensatz Example (Not Expanded out)

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land \; (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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$$(1-x)(1-z)$$
$$(1-y)z$$
$$(1-x)y(1-u)$$
$$yu$$
$$(1-u)(1-v)$$
$$xv$$
$$u(1-w)$$
$$xuw$$

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$$\begin{aligned}
& (1-y) \cdot (1-x)(1-z) \\
+\ & (1-x) \cdot (1-y)z \\
+\ & 1 \cdot (1-x)y(1-u) \\
+\ & (1-x) \cdot yu \\
+\ & x \cdot (1-u)(1-v) \\
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\end{aligned}$$

Size 27
Degree 3
(No use of Boolean axioms)

Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials q_i, r_j as unknowns
- Used successfully to solve, e.g., graph colouring problems [DLMM08, DLMO09, DLMM11]
- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

Dual Variables

- Annoying problem: $x_1 \lor x_2 \lor x_3$ translates to polynomial

$$(1-x_1)(1-x_2)(1-x_3) = 1-x_1-x_2-x_3+x_1x_2+x_1x_3+x_2x_3-x_1x_2x_3$$

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- More generally, exponential blow-up in # positive literals
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- Translate $C = \bigvee_{i \in \mathcal{P}} x_i \lor \bigvee_{j \in \mathcal{N}} \overline{x}_j$ to polynomial equations

$$\prod_{i \in \mathcal{P}} x_i' \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21] (also for other algebraic proof systems)

Polynomial Calculus [CEI96, ABRW02]

Nullstellensatz again

Infeasibility of

$$\begin{aligned} p_i(x_1, \ldots, x_n) &= 0 & i &\in [m] \\ x_j^2 - x_j &= 0 & j &\in [n] \\ x_j + x_j' - 1 &= 0 & j &\in [n] \end{aligned}$$

$$\Updownarrow$$

1 lies in polynomial ideal generated by these polynomials

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- $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x'_j - 1 \in \mathcal{I}$ for all $i \in [m]$, $j \in [n]$

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- If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$
- If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$

Polynomial Calculus Derivations and Refutations

- A polynomial calculus derivation is a sequence of polynomials in the ideal generated by p_i, $x_j^2 - x_j$, and $x_j + x_j' - 1$
- Derivation rules (from previous slide):
  - Axioms p_i, $x_j^2 - x_j$, and $x_j + x_j' - 1$
  - Linear combination $p, q \Rightarrow \alpha p + \beta q$
  - Monomial multiplication $p \Rightarrow m \cdot p$
- A refutation ends with the polynomial 1
- Complexity measures:
  - Size: total number of monomials in all polynomials in sequence expanded out
  - Degree: highest total degree of any polynomial
- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

Polynomial Calculus Can Simulate Resolution

Polynomial calculus can always simulate resolution proofs efficiently step by step

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Example: Resolution step

$$\frac{x \vee \overline{y} \vee z \qquad \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

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Example: Resolution step

$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

simulated by polynomial calculus derivation

$$\dfrac{x'yz' \qquad \dfrac{\dfrac{yz}{x'yz} \qquad \dfrac{z + z' - 1}{x'yz + x'yz' - x'y}}{-x'yz' + x'y}}{x'y}$$

Polynomial Calculus is Strictly Stronger than Resolution

Polynomial calculus can be exponentially stronger than resolution

For instance:

- Tseitin formulas on expander graphs if $\mathbb{F} = \mathrm{GF}(2)$
- Onto functional pigeonhole principle over any field [Rii93]

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Other hard formulas:

- Tseitin-like formulas for counting $\bmod p$ if $p \neq$ field characteristic [BGIP01]
- Random k-CNF formulas
    - all characteristics except 2 [BI99]
    - all characteristics [AR03]

Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials m, m', t:

1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

- Lexicographic
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Can write $p = \mathrm{lt}(p) + p'$ for $\mathrm{lt}(p)$ leading term (largest w.r.t. $\preceq$)

If $\mathrm{lt}(p) = t \cdot \mathrm{lt}(q)$, can reduce $p \bmod q$ by computing $p - t \cdot q$

"Multivariate division": Reduce p modulo all q in set of polynomials $\mathcal{G}$ until no further reductions possible

Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm (**very** rough)

1. Let $\mathcal{G} :=$ all axioms
2. Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
3. Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
4. Set $S := p' - q'$; reduce $S \bmod \mathcal{G}$ with multivariate division; add result to $\mathcal{G}$ if non-zero
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Computes so-called Gröbner basis

Fact: At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

Gröbner bases: Some Problems and Questions

1. Buchberger not a great SAT solving algorithm
   Slow and memory-intensive, and computes too much info
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   Possible to design dual-variable-aware Buchberger?!

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2. Dual variables increase reasoning power exponentially [dRLNS21]
   But are immediately eliminated by multivariate division
   Possible to design dual-variable-aware Buchberger?!

3. Analysis of polynomial calculus uses degree-lexicographic ordering
   In computational algebra, many other orderings used
   Prove proof complexity separation results for different orderings?

What About Algebraic SAT Solvers?

- Excitement about Gröbner basis approach after [CEI96]
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- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus
- Is it harder to build good algebraic SAT solvers, or is it just that too little work has been done (or both)?
- But very successful work on circuit verification in [KFB20, KB20, KBK20a, KBK20b, KB21, KBBN22]

SAT as System of 0-1 Integer Linear Inequalities

- Given CNF formula $F = \bigwedge_{i=1}^{m} C_i$

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$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

- Add variable axioms

$$x_j \geq 0$$
$$-x_j \geq -1$$

for all variables

Cutting Planes Proof System [CCT87]

Cutting planes introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

Can be applied to any system of 0-1 integer linear inequalities

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Cutting planes derivation rules

Multiplication $$\frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

Addition $$\frac{\sum a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

Division $$\frac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of 0-1 integer linear inequalities derived from
  - Axioms (clauses and variable bounds)
  - Multiplication $\sum a_i x_i \geq A \Rightarrow \sum c a_i x_i \geq cA$
  - Addition $\sum a_i x_i \geq A$, $\sum b_i x_i \geq B \Rightarrow \sum (a_i + b_i) x_i \geq A + B$
  - Division $\sum c a_i x_i \geq A \Rightarrow \sum a_i x_i \geq \lceil A/c \rceil$
- A refutation ends with the inequality $0 \geq 1$
- Complexity measures:
  - Length: $\#$ inequalities
  - Size: Count also bit size of representing all coefficients

Cutting Planes vs. Resolution

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- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger
  (e.g., for PHP, just count and argue that #pigeons > #holes)
- And 0-1 linear inequalities are similar to but much more concise than CNF

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
&(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\
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Hard Formulas for Cutting Planes

Clique-colouring formulas [Pud97]
"A graph with an m-clique is not $(m-1)$-colourable"

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Variables

- $p_{i,j}$ indicators of the edges in graph; $1 \leq i < j \leq n$
- $q_{k,i}$ identify members of m-clique; $1 \leq k \leq m$, $1 \leq i \leq n$
- $r_{i,\ell}$ specify colouring of vertices; $1 \leq \ell \leq m-1$, $1 \leq i \leq n$

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$q_{k,1} \vee q_{k,2} \vee \cdots \vee q_{k,n}$	some vertex is the kth member of clique
$\overline{q}_{k,i} \vee \overline{q}_{k',i}$	clique members are uniquely defined $(k \neq k')$
$p_{i,j} \vee \overline{q}_{k,i} \vee \overline{q}_{k',j}$	clique members are connected by edges
$r_{i,1} \vee r_{i,2} \vee \cdots \vee r_{i,m-1}$	every vertex i has a colour
$\overline{p}_{i,j} \vee \overline{r}_{i,\ell} \vee \overline{r}_{j,\ell}$	neighbours have distinct colours

More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses interpolation and circuit complexity

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
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Cutting planes not well understood at all
Clear need for development of new analysis methods
Some exciting contributions in [HP17, FPPR22, GGKS20]

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Lower bound for clique-colouring formulas uses interpolation and circuit complexity

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
- Hence, no small cutting planes proofs can exist either

Cutting planes not well understood at all
Clear need for development of new analysis methods
Some exciting contributions in [HP17, FPPR22, GGKS20]

Surprisingly, Tseitin formulas are at most quasi-polynomially hard for cutting planes [DT20]!

SAT Solvers Based on Cutting Planes?

So-called pseudo-Boolean (PB) solvers using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

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Challenge 1: Conjunctive normal form

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Is it truly harder to build good pseudo-Boolean solvers?
Or has just so much more work has been put into CDCL solvers?

Sherali-Adams (SA) and Sums of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) = 1$$

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Sums of squares (SoS) $(s_k \in \mathbb{R}[x_1, \ldots, x_n])$

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{s} s_k^2 = -1$$

SA, SoS, and Other Proof Systems

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Strict hierarchy (over $\mathbb{R}$):

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Sums of squares is strictly stronger than polynomial calculus (over $\mathbb{R}$) while Sherali-Adams and polynomial calculus are incomparable [Ber18]

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Sums of squares very strong proof system, except it cannot do parity reasoning efficiently [GV01, Gri01]

Survey [FKP19] is recommended for more reading

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Stabbing planes with polynomial-size coefficient can be simulated by cutting planes with quasi-polynomial overhead [DT20, FGI+21]

Extended Resolution [Tse68]

Resolution rule

$$\frac{C_1 \vee x \qquad C_2 \vee \overline{x}}{C_1 \vee C_2}$$

Extension rule introducing clauses

$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

for fresh variable a (encoding that $a \leftrightarrow (x \wedge y)$ must hold)

Extended Resolution and SAT Solving

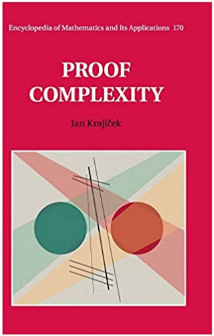
- Closely related (and equivalent) to *DRAT* proof system used to justify correctness of some preprocessing techniques [JHB12]
- *DRAT* also used for SAT solver proof logging
- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]
- Without restrictions, corresponds to extremely strong extended Frege system [CR79] — pretty much no lower bounds known
- To analyse solvers using extended resolution, would need to:
  - Describe heuristics/rules actually used
  - See if possible to reason about such restricted proof system

Some More References for Further Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

Summing up This Presentation

Overview of some proof systems used in combinatorial solving:

- Resolution $\longleftrightarrow$ DPLL and CDCL
- Nullstellensatz and polynomial calculus $\longleftrightarrow$ Gröbner bases
- Cutting planes $\longleftrightarrow$ pseudo-Boolean solving

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- Be a fun playground for theory-practice interaction!

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Thank you for your attention!

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