Towards an Optimal Separation of Space and Length in Resolution

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Executive Summary of Talk

- Resolution: proof system for refuting CNF formulas
- Perhaps *the* most studied system in proof complexity
- Basis of current state-of-the-art SAT-solvers (winners in SAT 2007 competition: resolution + clause learning)
- Key resources: time and space
- What are the connections between these resources?
  Are time and space correlated?
  Are there time/space trade-offs?

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \ldots \lor a_k$: disjunction of literals
  At most k literals: k-clause
- CNF formula $F = C_1 \land \ldots \land C_m$: conjunction of clauses
  k-CNF formula: CNF formula consisting of k-clauses
  (assume k fixed)
- Refer to clauses of CNF formula as axioms
  (as opposed to derived clauses)

Resolution Rule

Resolution rule:

$$\frac{B \vee x \qquad C \vee \overline{x}}{B \vee C}$$

Prove F unsatisfiable by deriving the unsatisfiable empty clause 0 (the clause with no literals) from F by resolution

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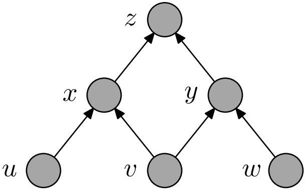
Example CNF Formula

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

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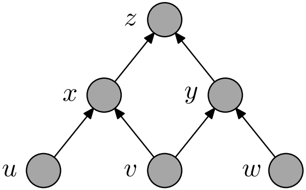

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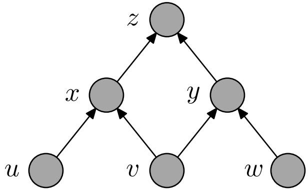

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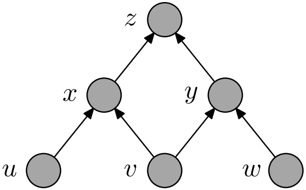

Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	0
# literals in largest clause	0
# lines on blackboard used	0

Can write down axioms,
erase used clauses or
infer new clauses (but only from clauses currently on the board!)

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	1
# literals in largest clause	1
# lines on blackboard used	1

u

Write down axiom 1: u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	2
# literals in largest clause	1
# lines on blackboard used	2

u
v

Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

u
v
$\overline{u} \vee \overline{v} \vee x$

Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

u
v
$\overline{u} \vee \overline{v} \vee x$
$\overline{v} \vee x$

Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Erase clause $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

u
v
$\overline{v} \lor x$

Erase clause $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

u
v
$\overline{v} \lor x$

Erase clause u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

v
$\overline{v} \lor x$

Erase clause u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

v
$\overline{v} \lor x$

Infer x from
v and $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

v
$\overline{v} \vee x$
x

Infer x from
v and $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

v
$\overline{v} \lor x$
x

Erase clause $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

v
x

Erase clause $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

v
x

Erase clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

x

Erase clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{x} \lor \overline{y} \lor z$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{x} \vee \overline{y} \vee z$
$\overline{y} \vee z$

Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase clause $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{y} \lor z$

Erase clause $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{y} \lor z$

Erase clause x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \vee z$

Erase clause x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee z$

Erase clause $\overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \vee \overline{w} \vee z$

Erase clause $\overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	10
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \lor \overline{w} \lor z$
v

Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	11
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \lor \overline{w} \lor z$
v
w

Write down axiom 3: w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Write down axiom 7: $\overline{z}$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$
$\overline{w} \vee z$

Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

w
$\overline{z}$
$\overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
# literals in largest clause	3
# lines on blackboard used	5

w
$\overline{z}$
$\overline{w} \lor z$

Infer z from
w and $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

w
$\overline{z}$
$\overline{w} \lor z$
z

Infer z from
w and $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
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w
$\overline{z}$
$\overline{w} \vee z$
z

Erase clause w

Example Resolution Refutation

1. u
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4. $\overline{u} \lor \overline{v} \lor x$
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Blackboard bookkeeping	
total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

$\overline{z}$
$\overline{w} \lor z$
z

Erase clause w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

$\overline{z}$
$\overline{w} \vee z$
z

Erase clause $\overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
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4. $\overline{u} \vee \overline{v} \vee x$
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Blackboard bookkeeping	
total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

$\overline{z}$

z

Erase clause $\overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
# literals in largest clause	3
# lines on blackboard used	5

$\overline{z}$
z

Infer 0 from
$\overline{z}$ and z

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	15
# literals in largest clause	3
# lines on blackboard used	5

$\overline{z}$
z
0

Infer 0 from
$\overline{z}$ and z

Length, Width and Space

- Length $L(\pi)$ of refutation $\pi : F \vdash 0$
  total # clauses in all of π
  (in our example 15)
- Width $W(\pi)$ of refutation $\pi : F \vdash 0$
  # literals in largest clause in π
  (in our example 3)
- Space $Sp(\pi)$ of refutation $\pi : F \vdash 0$
  max # clauses on blackboard simultaneously
  (in our example 5)

Length, Width and Space of Refuting F

- Length of refuting F is

$$L(F \vdash 0) = \min_{\pi : F \vdash 0} \{L(\pi)\}$$

- Width of refuting F is

$$W(F \vdash 0) = \min_{\pi : F \vdash 0} \{W(\pi)\}$$

- Space of refuting F is

$$Sp(F \vdash 0) = \min_{\pi : F \vdash 0} \{Sp(\pi)\}$$

Why Should We Care About These Measures?

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm
- Width: Intimately connected to length and space ☺

Results for Length and Width

Length

Haken (1985), Urquhart (1987): polynomial-size CNF formula families with exponential lower bounds on refutation length

Width

- Always $W(F \vdash 0) \leq \#$ variables in F
- Ben-Sasson & Wigderson (1999): strong correlation between length and width of refuting formula

Results for Width and Space

Always $Sp(F \vdash 0) \leq$ size of F

All space and width bounds for “the usual suspects” coincide!?

Theorem (Atserias & Dalmau 2003)

For any unsatisfiable k-CNF formula F it holds that

$$\textit{space } Sp(F \vdash 0) \geq \textit{width } W(F \vdash 0) - \mathcal{O}(1) \ .$$

Theorem (Nordström 2006)

There are k-CNF formula families $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with

- *refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and*
- *refutation space $Sp(F_n \vdash 0) = \Theta(\log n)$.*

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Connection Between Length and Space?

Current state of knowledge	
Length vs. width	strongly correlated
Width vs. space	separated
Length vs. space	???

- Small space $\Rightarrow$ short length (easy)
- But does short length imply small space?
- Or are there formulas with short, easy refutations that must require large space?

Mentioned as open problem in several papers
No consensus on what the “right answer” should be

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Towards an Optimal Separation of Space and Length

Theorem (Nordström & Håstad 2008)

There are k-CNF formula families $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with

- *refutation length* $L(F_n \vdash 0) = \mathcal{O}(n)$,
- *refutation width* $W(F_n \vdash 0) = \mathcal{O}(1)$ *and*
- *refutation space* $Sp(F_n \vdash 0) = \Theta(\sqrt{n})$.

Best separation of space and length so far

Exponential improvement of previous space-width separation

Any Practical Implications?

Yes and no

Space measures memory consumption for clause learning algorithms but space $\leq$ formula size—practical applications usually will have much more memory available than that

But maybe lower bounds on space can give clue about hardness anyway

(Sabharwal et al. 2003) exhibits formulas with very short refutations that state-of-the-art SAT-solver cannot find

Exactly the formulas in our $\Theta(\sqrt{n})$ space bound!

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(Sabharwal et al. 2003) exhibits formulas with very short refutations that state-of-the-art SAT-solver cannot find

Exactly the formulas in our $\Theta(\sqrt{n})$ space bound!

How to Separate Length and Space?

Want to find formulas that

- can be quickly refuted
- but require large space

Such time-space trade-off questions well-studied for pebble games modelling calculations described by DAGs

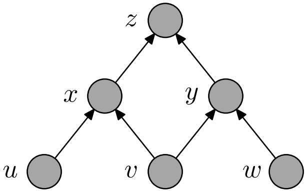
- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

Known result: ∃ DAGs requiring many pebbles in terms of size

Look at CNF formulas encoding pebbles games on DAGs!

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G



Number of pebbles	
Current	0
Max so far	0

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

The Black-White Pebble Game

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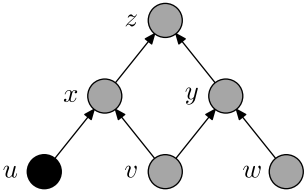
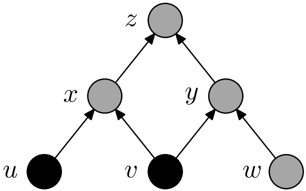


Number of pebbles	
Current	1
Max so far	1

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of *G*

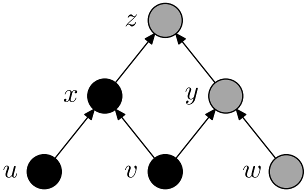


Number of pebbles	
Current	2
Max so far	2

1. Can place black pebble on (empty) vertex *v* if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from *v* if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

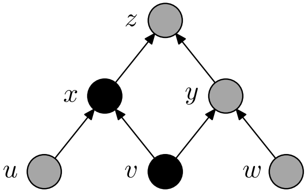


Number of pebbles	
Current	3
Max so far	3

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

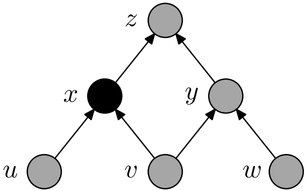


Number of pebbles	
Current	2
Max so far	3

1. Can place black pebble on (empty) vertex *v* if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from *v* if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

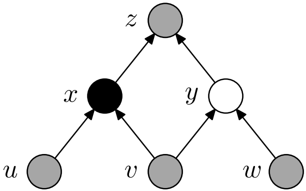


Number of pebbles	
Current	1
Max so far	3

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

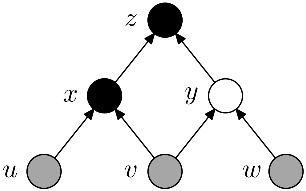


Number of pebbles	
Current	2
Max so far	3

1. Can place black pebble on (empty) vertex *v* if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from *v* if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

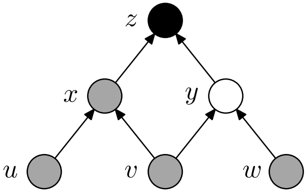


Number of pebbles	
Current	3
Max so far	3

1. Can place black pebble on (empty) vertex *v* if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from *v* if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G



Number of pebbles	
Current	2
Max so far	3

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
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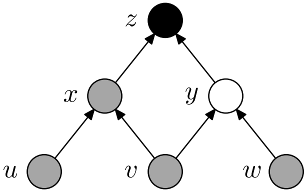

Number of pebbles	
Current	2
Max so far	3

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
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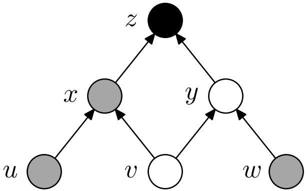
Goal: get single black pebble on sink vertex of *G*

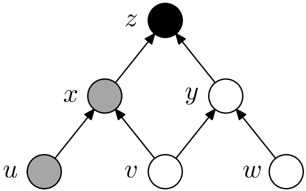


Number of pebbles	
Current	3
Max so far	3

1. Can place black pebble on (empty) vertex *v* if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from *v* if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

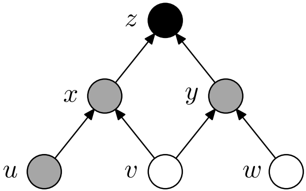


Number of pebbles	
Current	4
Max so far	4

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

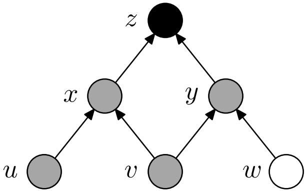


Number of pebbles	
Current	3
Max so far	4

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

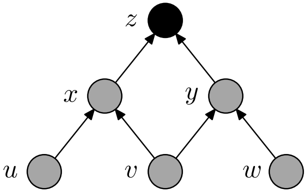


Number of pebbles	
Current	2
Max so far	4

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G



Number of pebbles	
Current	1
Max so far	4

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

Black-White Pebbling Price

- Cost of pebbling:
  max # pebbles simultaneously in G
  (in our example 4)
- Black-white pebbling price $\mathit{BW\text{-}Peb}(G)$ of DAG G:
  minimal cost of any pebbling
- Many bounds on pebbling price known
  E.g. pyramids of height h require $\Theta(h)$ pebbles

Pebbling Contradiction

CNF formula encoding pebble game on DAG G

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
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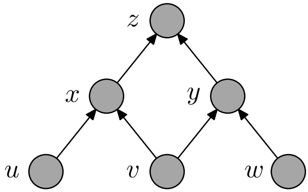

- sources are true
- truth propagates upwards
- but sink is false

Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling contradictions

To make this work, need more than one variable per vertex (but structure of formula is the same)

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CNF formula encoding pebble game on DAG G

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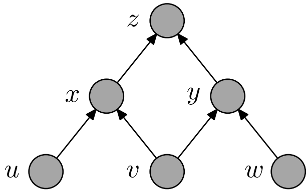

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Rephrasing Our Result

Theorem (Nordström & Håstad 2008)

The space of refuting pebbling contradictions with at least 2 variables per vertex over pyramids of height h is $\Theta(h)$.

Previously stated theorem follows as corollary since

- height $= \sqrt{\text{pyramid size}}$
- pebbling contradictions can be refuted in linear length and constant width (Ben-Sasson et al. 2000)

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Proof Idea

Resolution	Pebbling
Translate sets of clauses...	into black and white pebbles
...then the clause set must contain at least N clauses	Prove that if the translation results in N pebbles...
Show that consecutive sets of clauses on blackboard in a resolution refutation...	translates into a black-white pebbling of DAG corresponding to formula
...yielding same lower bound on space in resolution	Plug in lower bound on black-white pebbling price...

Proof Idea

Resolution	**Pebbling**
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Proof Idea

Resolution	**Pebbling**
Translate sets of clauses...	into black and white pebbles
...then the clause set must contain at least N clauses	Prove that if the translation results in N pebbles...
Show that consecutive sets of clauses on blackboard in a resolution refutation...	translates into a black-white pebbling of DAG corresponding to formula
...yielding same lower bound on space in resolution	Plug in lower bound on black-white pebbling price...

Interpreting Clauses in Terms of Pebbles

Black-white pebbling models non-deterministic computation

- black pebbles $\Leftrightarrow$ computed results
- white pebbles $\Leftrightarrow$ guesses needing to be verified

"We know z assuming v, w"

Corresponds to that "blackboard implies z true if we also assume v and w true"

This is the case e.g. for blackboard $\boxed{\overline{v} \lor \overline{w} \lor z}$ derived from example formula

Interpreting Clauses in Terms of Pebbles

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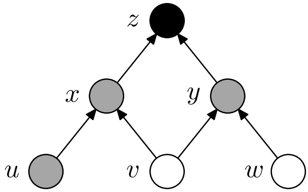

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Interpreting Clauses in Terms of Pebbles

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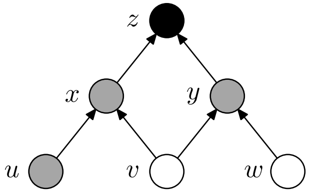


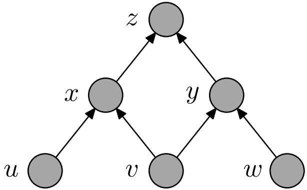
"We know z assuming v, w"

Corresponds to that "blackboard implies z true if we also assume v and w true"

This is the case e.g. for blackboard $\boxed{\overline{v} \lor \overline{w} \lor z}$ derived from example formula

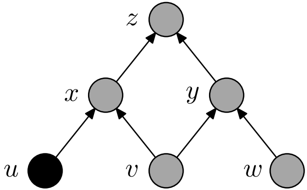
Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

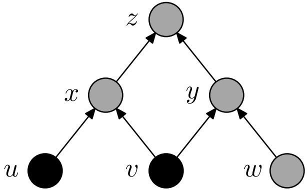


u

Write down axiom 1: u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

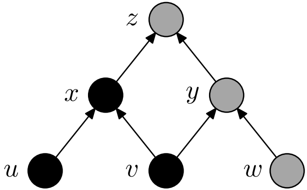


u
v

Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

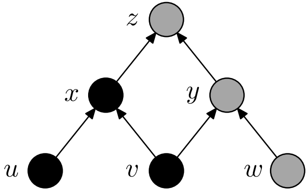


u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

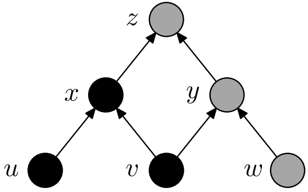


u
v
$\overline{u} \lor \overline{v} \lor x$

Infer $\overline{v} \lor x$ from u and $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

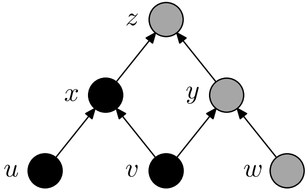


u
v
$\overline{u} \vee \overline{v} \vee x$
$\overline{v} \vee x$

Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

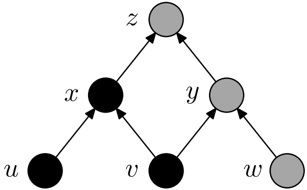


u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Erase clause $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

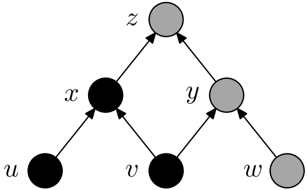


u
v
$\overline{v} \lor x$

Erase clause $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

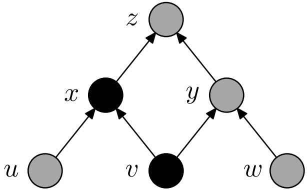


u
v
$\overline{v} \vee x$

Erase clause u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

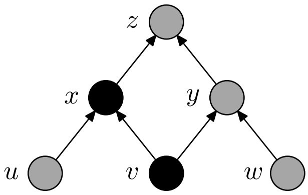


v
$\overline{v} \vee x$

Erase clause u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

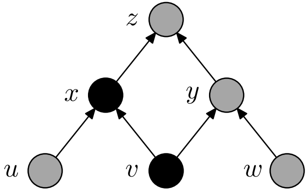


v
$\overline{v} \lor x$

Infer x from
v and $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

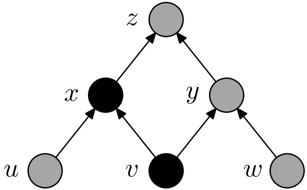


v
$\overline{v} \vee x$
x

Infer x from
v and $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

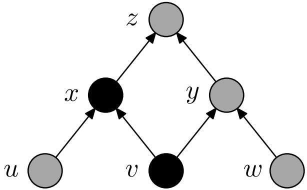


v
$\overline{v} \vee x$
x

Erase clause $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

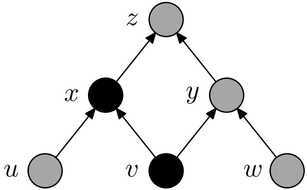


v
x

Erase clause $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

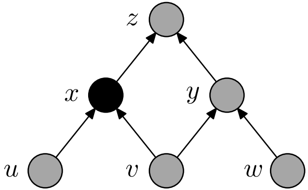


v
x

Erase clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

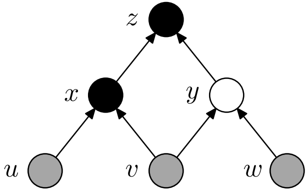


x

Erase clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

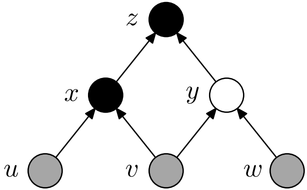


x
$\overline{x} \vee \overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

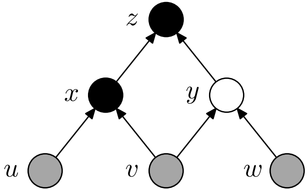


x
$\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from x and $\overline{x} \vee \overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

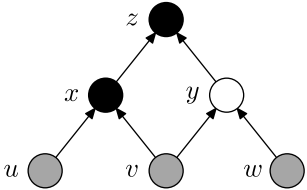


x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

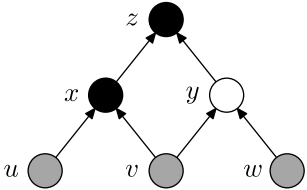


x
$\overline{x} \vee \overline{y} \vee z$
$\overline{y} \vee z$

Erase clause $\overline{x} \vee \overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

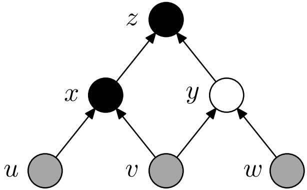


x
$\overline{y} \lor z$

Erase clause $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

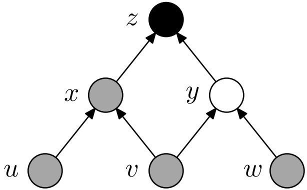


x
$\overline{y} \vee z$

Erase clause x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

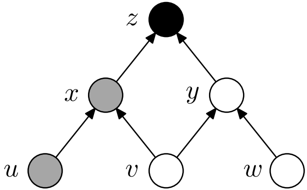


$\overline{y} \lor z$

Erase clause x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

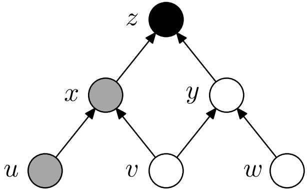


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

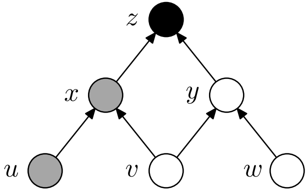


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

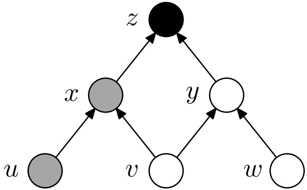


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$
$\overline{v} \vee \overline{w} \vee z$

Infer $\overline{v} \vee \overline{w} \vee z$ from $\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

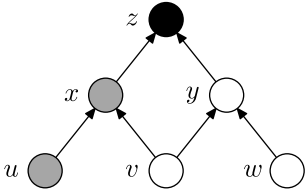


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$
$\overline{v} \vee \overline{w} \vee z$

Erase clause $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

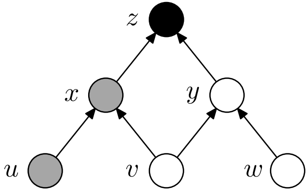


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

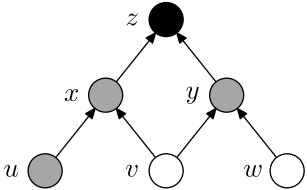


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

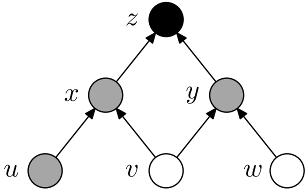


$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

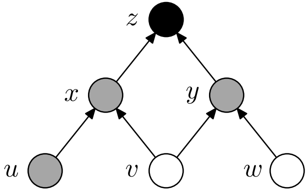


$\overline{v} \vee \overline{w} \vee z$
v

Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

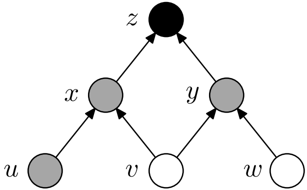


$\overline{v} \vee \overline{w} \vee z$
v
w

Write down axiom 3: w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

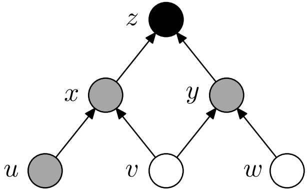


$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Write down axiom 7: $\overline{z}$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

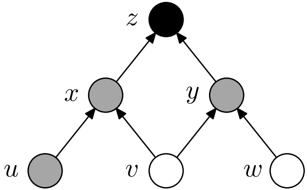


$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$

Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

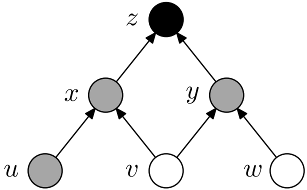


$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$
$\overline{w} \vee z$

Infer $\overline{w} \vee z$ from v and $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

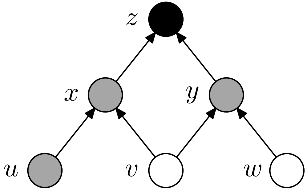


$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

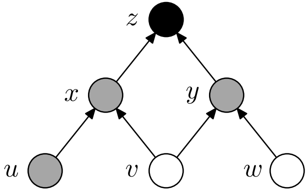


$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

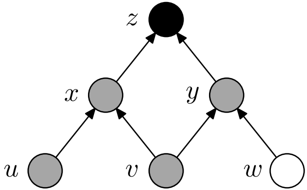


$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

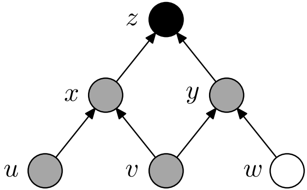


w
$\overline{z}$
$\overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

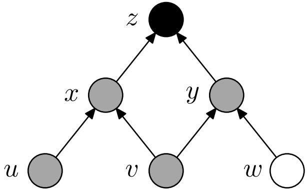


w
$\overline{z}$
$\overline{w} \vee z$

Infer z from
w and $\overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



w
$\overline{z}$
$\overline{w} \vee z$
z

Infer z from
w and $\overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
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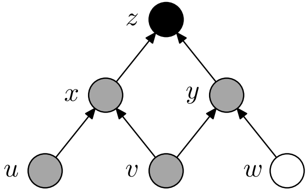
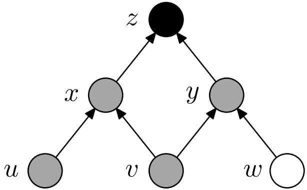


w
$\overline{z}$
$\overline{w} \vee z$
z

Erase clause w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

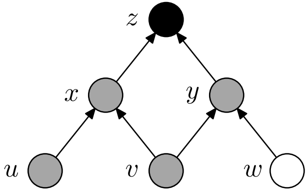


$\overline{z}$
$\overline{w} \vee z$
z

Erase clause w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

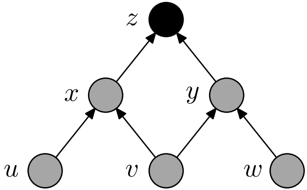


$\overline{z}$
$\overline{w} \vee z$
z

Erase clause $\overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

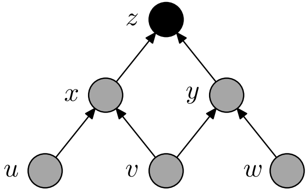


$\overline{z}$
z

Erase clause $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

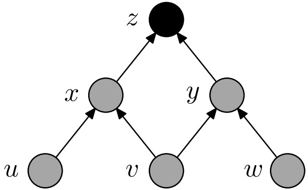


$\overline{z}$
z

Infer 0 from
$\overline{z}$ and z

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



$\overline{z}$
z
0

Infer 0 from
$\overline{z}$ and z

Sweeping the details under the rug. . .

This looks very nice, but in reality things get (much) messier

Refutations have no reason to derive nicely structured clauses
$\Rightarrow$ cannot extract pebblings from refutations

Different ideas needed

But this is the guiding intuition behind the proof

Separating Space and Length Optimally

Only able to prove our construction for restricted class of DAGs

Proof for general DAGs would imply separation of space and length with length $\mathcal{O}(n)$ and space $\Omega(n/\log n)$

Would be optimal—given length n, always possible to achieve space $\mathcal{O}(n/\log n)$

Theorem (Ben-Sasson & Nordström, March 2008)

There are k-CNF formula families $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with

- *refutation length $L(F_n \vdash 0) = \mathcal{O}(n)$,*
- *refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and*
- *refutation space $Sp(F_n \vdash 0) = \Omega(n/\log n)$.*

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- *refutation width* $W(F_n \vdash 0) = \mathcal{O}(1)$ *and*
- *refutation space* $Sp(F_n \vdash 0) = \Omega(n/\log n)$*.*

Prove Space-Length Trade-offs

Open Question

Are there formulas refutable in short length and small space, but for which any small-space refutation must be long?

We are currently working on this. . .

Answer seems to be yes, possibly in a very strong sense

Could be bad news for proof search algorithms

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Are there formulas refutable in short length and small space, but for which any small-space refutation must be long?

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Conclusions

- This work: Space-length separation with formulas refutable in length $\mathcal{O}(n)$ and space $\Omega(\sqrt{n})$
- More recently: Optimal separation with formulas refutable in length $\mathcal{O}(n)$ and space $\Omega(n/\log n)$
- Ongoing work: Trade-offs between space and length
  Some results but a number of open problems remain

Thank you for your attention!