Understanding the Hardness of Proving Formulas in Propositional Logic

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Based on joint work with Eli Ben-Sasson

A Fundamental Theoretical Problem...

Problem

Given a propositional logic formula F, is it true no matter how we assign values to its variables?

TAUTOLOGY: Fundamental problem in theoretical computer science ever since Stephen Cook's NP-completeness paper in 1971

Also posed as one of the main challenges for all of mathematics in the new millennium by the Clay Mathematics Institute

Widely believed intractable in worst case — deciding whether this is so is one of the famous million dollar Millennium Problems

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. . . with Huge Practical Implications

- All known algorithms run in exponential time in worst case
- But enormous progress on applied computer programs last 10-15 years
- These so-called SAT solvers are routinely deployed to solve large-scale real-world problems with millions of variables
- Used in e.g. hardware verification, software testing, software package management, artificial intelligence, cryptography, bioinformatics, and more
- But we also know small example formulas with only hundreds of variables that trip up even state-of-the-art SAT solvers

What Makes Formulas Hard or Easy?

- Best known algorithms based on simple DPLL method (Davis-Putnam-Logemann-Loveland) from 1960s (although with many clever optimizations)
- How can these SAT solvers be so good in practice? And how can one determine whether a particular formula is tractable or too difficult?
- Key bottlenecks for SAT solvers: time and memory
- What are the connections between these resources? Are they correlated? Are there trade-offs?
- This talk: What can the field of proof complexity say about these questions?

Outline

1. SAT solving and Proof Complexity
   - Tautologies and CNF formulas
   - SAT solving and DPLL
   - Proof Complexity and Resolution
2. Time and Space Bounds and Trade-offs
   - Previous Work
   - Our Results
   - Some Proof Ingredients
3. Open Problems
   - Total Space in Resolution
   - Space in Stronger Proof Systems
   - Space and SAT solving

What Is a Tautology?

A tautological formula, or tautology, evaluates to true no matter how the variables are assigned values ($1 =$ true or $0 =$ false)

Example: "if x implies y, then not y implies not x, and vice versa"

In symbolic notation: $(x \to y) \leftrightarrow (\neg y \to \neg x)$

Verification by truth table:

x	y	$x \to y$	$\neg y \to \neg x$	$(x \to y) \leftrightarrow (\neg y \to \neg x)$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
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Non-example: $(x \to y) \leftrightarrow (y \to x)$
False for e.g. $x = 0$ and $y = 1$, so not a tautology

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Tautologies and CNF Formulas

Conjunctive normal form (CNF)

ANDs of ORs of variables or negated variables
(or conjunctions of disjunctive clauses)

Example:

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land\ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

Proving that a formula in propositional logic is **always** satisfied

$\Updownarrow$

Proving that a CNF formula is **never** satisfied
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Transforming Tautologies to Unsatisfiable CNF Formulas

- Introduce auxilary variables x_P, x_Q for all subformulas P, Q
- Write down clauses enforcing subformulas computed correctly
  E.g. for $F := P \rightarrow Q$ we get

$$(\neg x_P \lor x_Q \lor \neg x_F) \land (x_P \lor x_F) \land (\neg x_Q \lor x_F)$$

- Add clause $\neg x_F$ requiring whole formula F to evaluate to false

Then this CNF formula

- is unsatisfiable iff original formula tautology
- has essentially same size as original formula

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Some Terminology

- Literal a: variable x or its negation (from now on write $\overline{x}$ instead of $\neg x$)
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
(Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
(assume k fixed)
- Refer to clauses of CNF formula as axioms
(as opposed to derived clauses)

The DPLL Method

Based on [Davis & Putnam '60] and [Davis, Logemann & Loveland '62]

Somewhat simplified description:

- If F contains an empty clause (without literals), then report "unsatisfiable"
- Otherwise pick some variable x in F
- Set $x = 0$, simplify F and try to refute recursively
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$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when falsfied clause found

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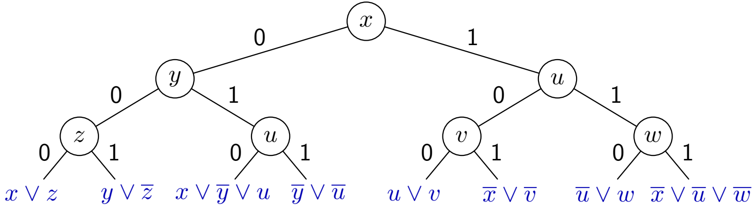

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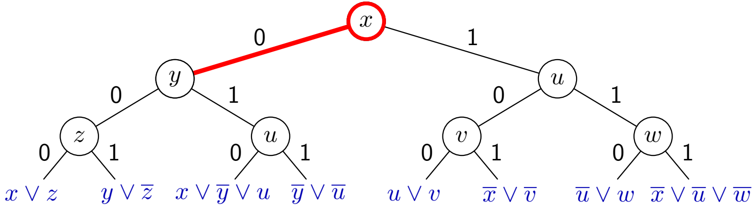

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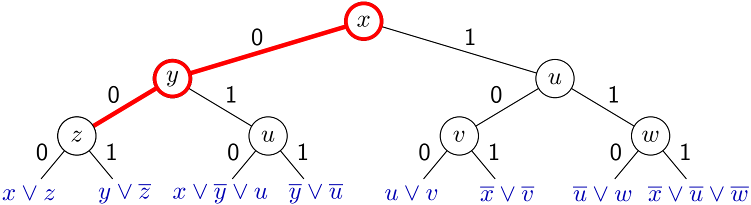

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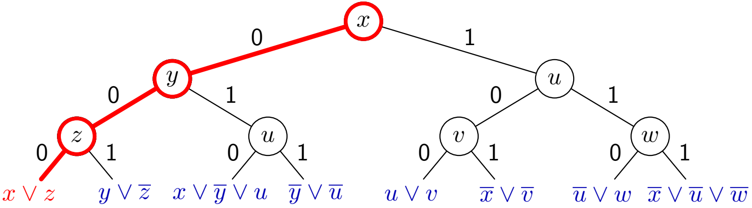

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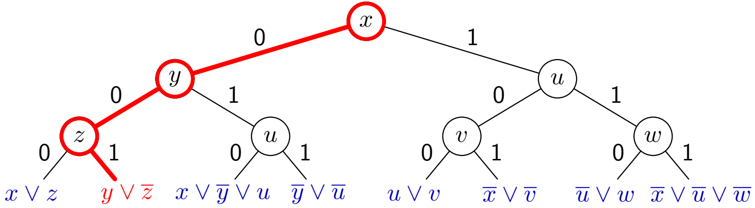

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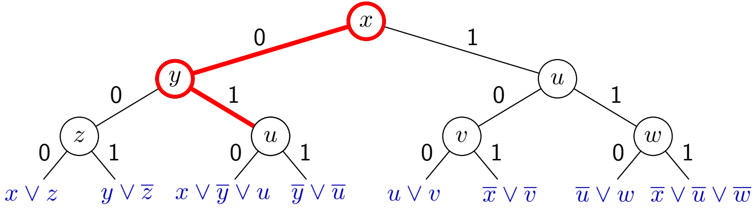

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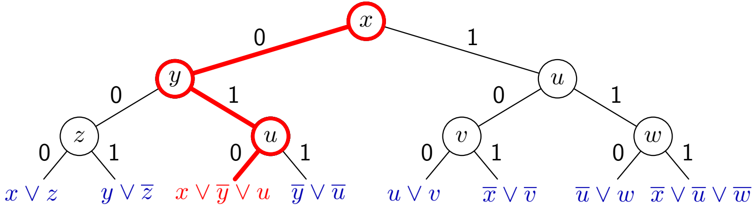

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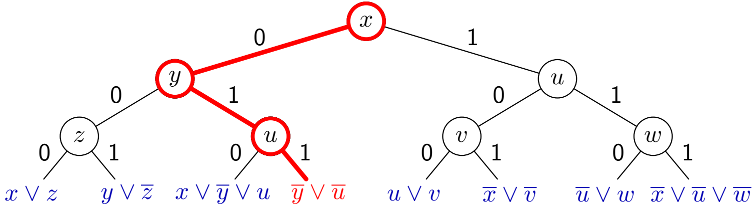

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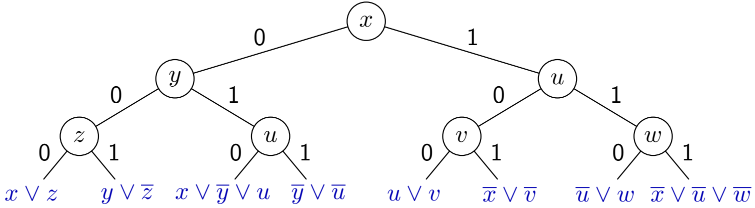

State-of-the-art DPLL SAT solvers

Many more ingredients in modern SAT solvers, for instance:

- Choice of pivot variables crucial
- When reaching falsified clause, compute why partial assignment failed — add this info to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)

Proof Complexity

Proof search algorithm: defines proof system with derivation rules

Proof complexity: study of proofs in such systems

- Lower bounds: no algorithm can do better (even optimal one always guessing the right move)
- Upper bounds: gives hope for good algorithms if we can search for proofs in system efficiently

Resolution

Resolution rule:

$$\frac{B \lor x \quad C \lor \overline{x}}{B \lor C}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $C_1, C_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

Prove F unsatisfiable by deriving the unsatisfiable empty clause 0 from F by resolution

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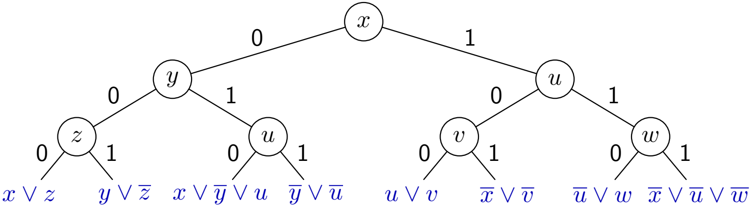
DPLL and Resolution

A DPLL execution is essentially a resolution proof

Look at our example again



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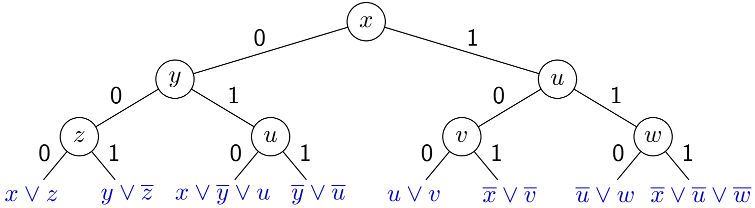

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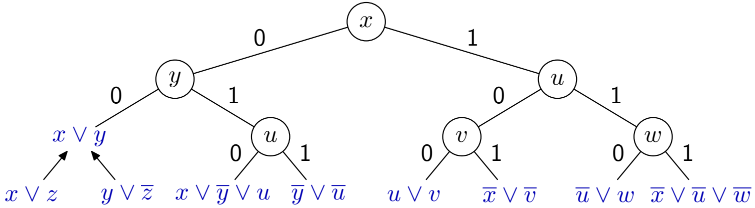

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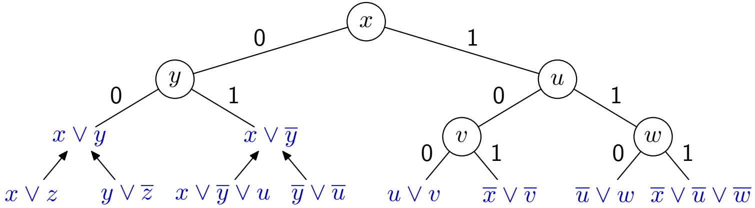

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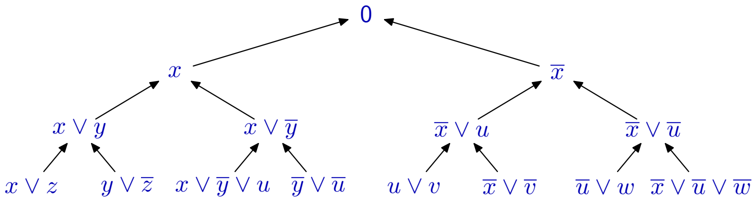

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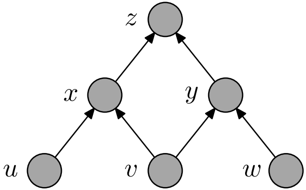
The Theoretical Model

- Goal: Refute given CNF formula (i.e., prove it is unsatisfiable)
- Proof system operates with disjunctive clauses
- Proof/refutation is "presented on blackboard"
- Derivation steps:
  - Write down clauses of CNF formula being refuted (axiom clauses)
  - Infer new clauses by resolution rule
  - Erase clauses that are not currently needed (to save space on blackboard)
- Refutation ends when empty clause 0 is derived

Example CNF Formula

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



Defined in terms of directed acyclic graph (DAG):

- source vertices true
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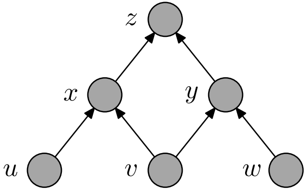

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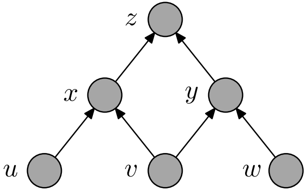

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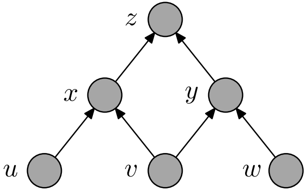
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	0
max # lines on board	0
max # literals on board	0

Can write down axioms, erase used clauses or infer new clauses by resolution rule

$$\frac{B \lor x \quad C \lor \overline{x}}{B \lor C}$$

(but only from clauses currently on the board!)

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	1
max # lines on board	1
max # literals on board	1

u

Write down axiom 1: u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	2
max # lines on board	2
max # literals on board	2

u
v

Write down axiom 1: u
Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
max # lines on board	3
max # literals on board	5

u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
max # lines on board	3
max # literals on board	5

u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

Board:

u
v
$\overline{v} \lor x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u
v
$\overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$
Erase the clause u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

v
$\overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$
Erase the clause u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

v

$\overline{v} \vee x$

u and $\overline{u} \vee \overline{v} \vee x$

Erase the clause $\overline{u} \vee \overline{v} \vee x$

Erase the clause u

Infer x from

v and $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

v
$\overline{v} \vee x$
x

u and $\overline{u} \vee \overline{v} \vee x$
Erase the clause $\overline{u} \vee \overline{v} \vee x$
Erase the clause u
Infer x from
v and $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

v
$\overline{v} \lor x$
x

Erase the clause $\overline{u} \lor \overline{v} \lor x$
Erase the clause u
Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

v
x

Erase the clause $\overline{u} \lor \overline{v} \lor x$
Erase the clause u
Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

v
x

Erase the clause u
Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$
Erase the clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

x

Erase the clause u
Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$
Erase the clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
max # lines on board	4
max # literals on board	7

x

$\overline{x} \lor \overline{y} \lor z$

Infer x from

v and $\overline{v} \lor x$

Erase the clause $\overline{v} \lor x$

Erase the clause v

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
max # lines on board	4
max # literals on board	7

x

$\overline{x} \lor \overline{y} \lor z$

Erase the clause $\overline{v} \lor x$
Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the clause $\overline{v} \lor x$
Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x
$\overline{y} \lor z$

Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x
$\overline{y} \lor z$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$
Erase the clause x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

$\overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$
Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the clause $\overline{x} \vee \overline{y} \vee z$
Erase the clause x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
max # lines on board	4
max # literals on board	7

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$
Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
max # lines on board	4
max # literals on board	7

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Erase the clause $\overline{x} \lor \overline{y} \lor z$
Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the clause $\overline{x} \lor \overline{y} \lor z$
Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{v} \vee \overline{w} \vee z$

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	10
max # lines on board	4
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v

Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$
Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	11
max # lines on board	4
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$
Write down axiom 2: v
Write down axiom 3: w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
max # lines on board	4
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$
Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
max # lines on board	4
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

w
$\overline{z}$
$\overline{w} \lor z$

v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

w
$\overline{z}$
$\overline{w} \lor z$
z

v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
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4. $\overline{u} \lor \overline{v} \lor x$
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7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

w
$\overline{z}$
$\overline{w} \lor z$
z

Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the clause w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$
$\overline{w} \lor z$
z

Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the clause w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$

$\overline{w} \lor z$

z

Erase the clause $\overline{v} \lor \overline{w} \lor z$

Infer z from

w and $\overline{w} \lor z$

Erase the clause w

Erase the clause $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$

z

Erase the clause $\overline{v} \vee \overline{w} \vee z$

Infer z from

w and $\overline{w} \vee z$

Erase the clause w

Erase the clause $\overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$

z

w and $\overline{w} \lor z$

Erase the clause w

Erase the clause $\overline{w} \lor z$

Infer 0 from

$\overline{z}$ and z

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	15
max # lines on board	5
max # literals on board	8

$\overline{z}$
z
0

w and $\overline{w} \lor z$
Erase the clause w
Erase the clause $\overline{w} \lor z$
Infer 0 from
$\overline{z}$ and z

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

\# clauses written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

x
$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$

Clause space: 3

Total space: 6

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$\#$ clauses written on blackboard counted with repetitions

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$\overline{y} \lor z$
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Length
clauses written on blackboard counted with repetitions

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Somewhat less straightforward — several ways of measuring

1. x
2. $\overline{y} \lor z$
3. $\overline{v} \lor \overline{w} \lor y$

Clause space: 3

Total space: 6

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

clauses written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

$$x^{1}$$
$$\overline{y}^{2} \lor z^{3}$$
$$\overline{v}^{4} \lor \overline{w}^{5} \lor y^{6}$$

Clause space: 3

Total space: 6

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

\# clauses written on blackboard counted with repetitions
(in our example resolution refutation 15)

Space

Somewhat less straightforward — several ways of measuring

$$x$$
$$\overline{y} \lor z$$
$$\overline{v} \lor \overline{w} \lor y$$

Clause space: 3
(in our refutation 5)
Total space: 6
(in our refutation 8)

Length and Space Bounds for Resolution

Let $n =$ size of formula

Length: at most 2^n
Matching lower bound up to constant factors in exponent [Urquhart '87, Chvátal & Szemerédi '88]

Clause space: at most n
Matching lower bound up to constant factors [Torán '99, Alekhnovich et al. '00]

Total space: at most n^2
No better lower bound than linear in n!?

[Sidenote: space bounds hold even for "magic algorithms" always making optimal choices — so might be much stronger in practice]

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Comparing Length and Space

Some "rescaling" needed to get meaningful comparisons of length and space

- Length exponential in formula size in worst case
- Clause space at most linear
- So natural to compare space to logarithm of length

Length-Space Correlations and/or Trade-offs?

$\exists$ constant space refutation $\Rightarrow$ $\exists$ polynomial length refutation [Atserias & Dalmau '03]

Does short length imply small space?
Has been open — even no consensus on likely "right answer"

Essentially nothing known about length-space trade-offs for resolution refutations in the general, unrestricted proof system

(Some trade-off results in restricted settings in [Ben-Sasson '02, Nordström '07])

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Our results 1: An Optimal Length-Space Separation

Length and space in resolution are "completely uncorrelated"

Theorem

There are k-CNF formula families of size n with

- *refutation length linear in n requiring*
- *clause space growing like $n/\log n$*

Optimal separation of length and space — given length n, always possible to achieve clause space $\approx n/\log n$ (within constant factors)

Our Results 2: Length-Space Trade-offs

We prove collection of length-space trade-offs

Results hold for

- resolution
- even stronger proof systems (which we won't go into here)

Different trade-offs covering (almost) whole range of space from constant to linear

Simple, explicit formulas

One Example: Robust Trade-offs for Small Space

Theorem

For any arbitrarily slowly growing function g there exist explicit CNF formulas of size n

- *refutable in space $g(n)$ and*
- *refutable in length linear in n and space $\approx \sqrt[3]{n}$ such that*
- *any resolution refutation in space $\ll \sqrt[3]{n}$ requires superpolynomial length*

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How to Get a Handle on Time-Space Relations?

Questions about time-space trade-offs fundamental in theoretical computer science

In particular, well-studied (and well-understood) for pebble games modelling calculations described by DAGs ([Cook & Sethi '76] and many others)

- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

Some quick graph terminology

- DAGs consist of vertices with directed edges between them
- vertices with no incoming edges: sources
- vertices with no outgoing edges: sinks

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Some quick graph terminology

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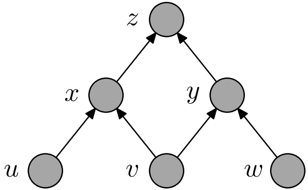
The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



# moves	0
Current # pebbles	0
Max # pebbles so far	0

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
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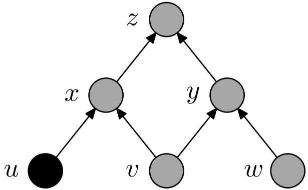

# moves	1
Current # pebbles	1
Max # pebbles so far	1

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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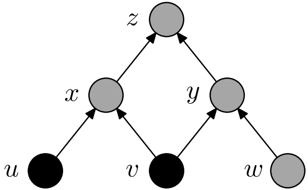

# moves	2
Current # pebbles	2
Max # pebbles so far	2

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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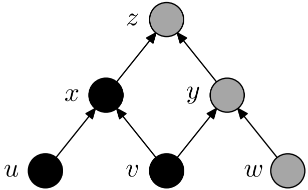

# moves	3
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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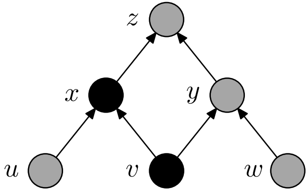

# moves	4
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
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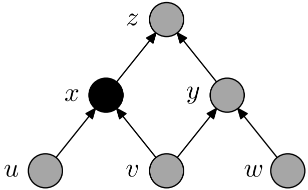
The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



# moves	5
Current # pebbles	1
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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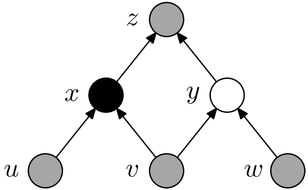
Goal: get single black pebble on sink vertex z of G



# moves	6
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all predecessors have pebbles

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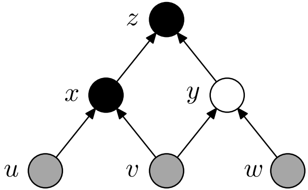
Goal: get single black pebble on sink vertex z of G



# moves	7
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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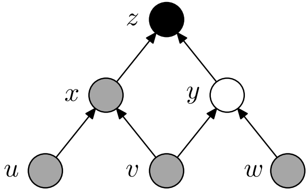

# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
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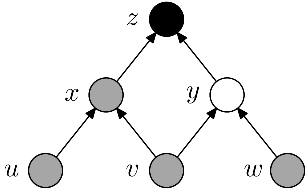

# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
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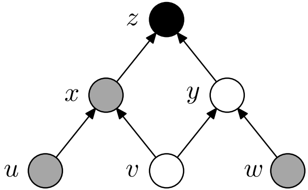

# moves	9
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
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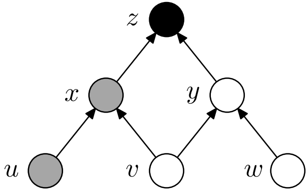
Goal: get single black pebble on sink vertex z of G



# moves	10
Current # pebbles	4
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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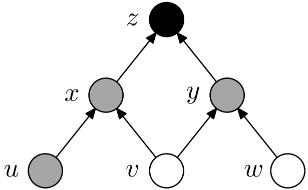

# moves	11
Current # pebbles	3
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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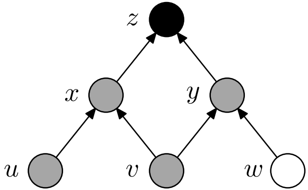

# moves	12
Current # pebbles	2
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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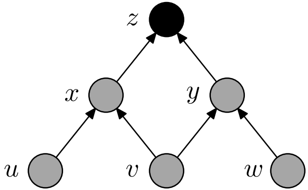


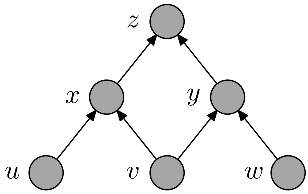
# moves	13
Current # pebbles	1
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all predecessors have pebbles

Pebbling Contradiction

CNF formula encoding pebble game on DAG G

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



- sources are true
- truth propagates upwards
- but sink is false

Studied by [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and others

Our hope is that pebbling properties of DAG somehow carry over to resolution refutations of pebbling contradictions

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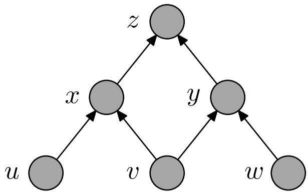

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Interpreting Refutations as Black-White Pebblings

Black-white pebbling models non-deterministic computation (where one can guess partial results and verify later)

- black pebbles ⇔ computed results
- white pebbles ⇔ guesses needing to be verified

"Know z assuming v, w"

Corresponds to $(v \wedge w) \rightarrow z$, i.e., blackboard clause $\overline{v} \vee \overline{w} \vee z$

So translate clauses to pebbles by:
unnegated variable ⇒ black pebble
negated variable ⇒ white pebble

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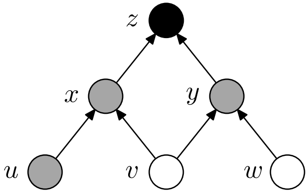

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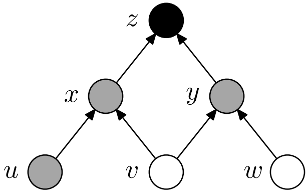

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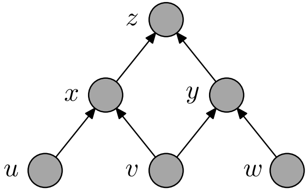
unnegated variable ⇒ black pebble
negated variable ⇒ white pebble

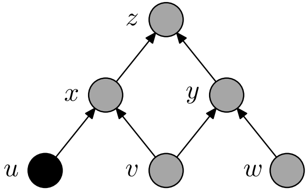
Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

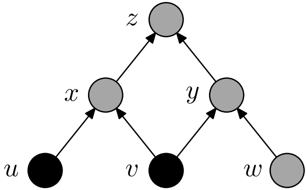


u

Write down axiom 1: u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

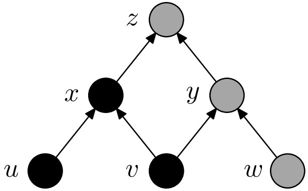


u
v

Write down axiom 1: u
Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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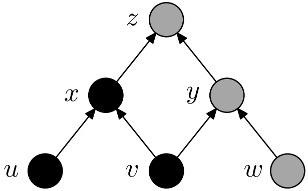

---

u

v

$\overline{u} \lor \overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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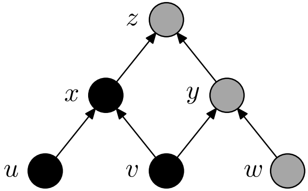


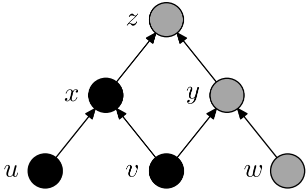
---

u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



---

u

v

$\overline{u} \vee \overline{v} \vee x$

$\overline{v} \vee x$

Write down axiom 2: v

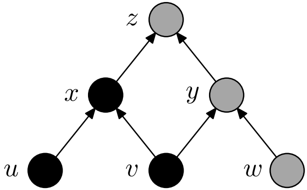
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Erase the clause $\overline{u} \vee \overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



---

u

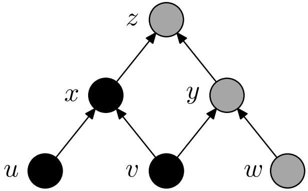
v

$\overline{v} \vee x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$
Erase the clause $\overline{u} \vee \overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

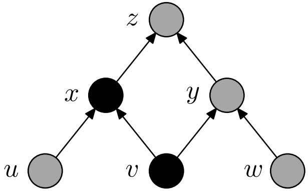


u
v
$\overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$
Erase the clause u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



v

$\overline{v} \vee x$

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Infer $\overline{v} \vee x$ from

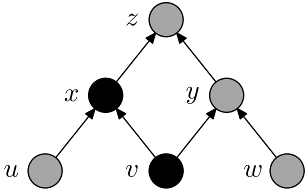
u and $\overline{u} \vee \overline{v} \vee x$

Erase the clause $\overline{u} \vee \overline{v} \vee x$

Erase the clause u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



---

v
$\overline{v} \lor x$

u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$
Erase the clause u
Infer x from
v and $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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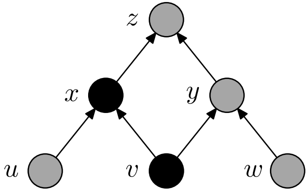

---

v

$\overline{v} \lor x$

x

u and $\overline{u} \lor \overline{v} \lor x$

Erase the clause $\overline{u} \lor \overline{v} \lor x$

Erase the clause u

Infer x from

v and $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
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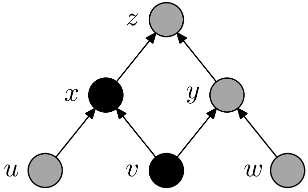
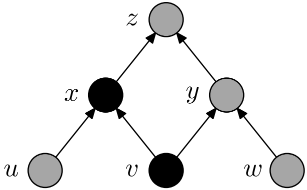


v
$\overline{v} \lor x$
x

Erase the clause $\overline{u} \lor \overline{v} \lor x$
Erase the clause u
Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



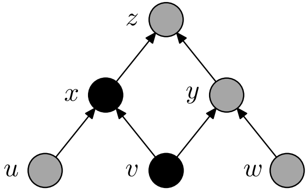
---

v
x

Erase the clause $\overline{u} \vee \overline{v} \vee x$
Erase the clause u
Infer x from
v and $\overline{v} \vee x$
Erase the clause $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



---

v
x

Erase the clause u
Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$
Erase the clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
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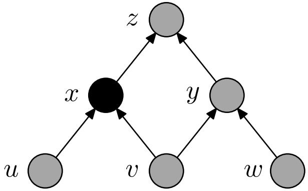
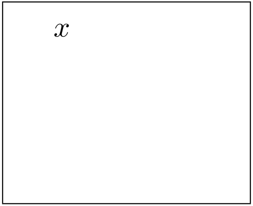
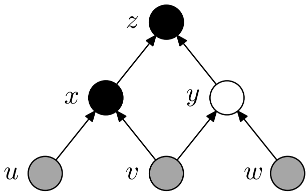




Erase the clause u
Infer x from
v and $\overline{v} \lor x$
Erase the clause $\overline{v} \lor x$
Erase the clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



x

$\overline{x} \lor \overline{y} \lor z$

Infer x from

v and $\overline{v} \lor x$

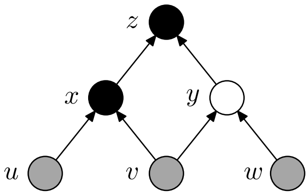
Erase the clause $\overline{v} \lor x$

Erase the clause v

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

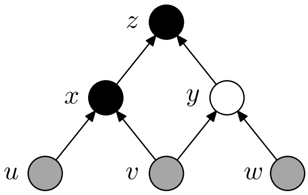


x
$\overline{x} \lor \overline{y} \lor z$

Erase the clause $\overline{v} \lor x$
Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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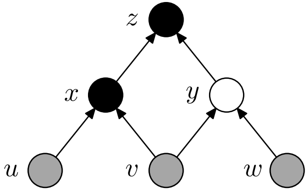
x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the clause $\overline{v} \lor x$
Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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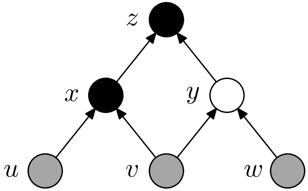
7. $\overline{z}$



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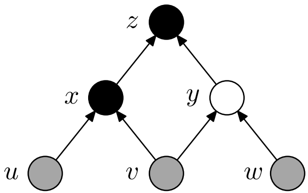
x

$\overline{y} \lor z$

Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



---

x

$\overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from

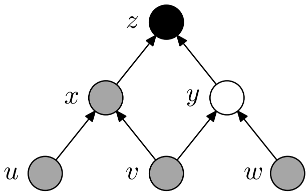
x and $\overline{x} \vee \overline{y} \vee z$

Erase the clause $\overline{x} \vee \overline{y} \vee z$

Erase the clause x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

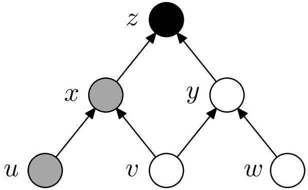


$\overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$
Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the clause $\overline{x} \vee \overline{y} \vee z$
Erase the clause x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

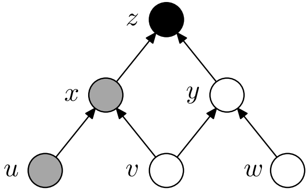


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$

Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the clause $\overline{x} \vee \overline{y} \vee z$
Erase the clause x
Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

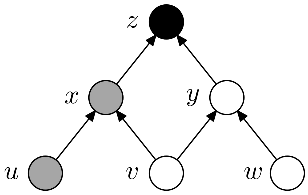


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$

Erase the clause $\overline{x} \vee \overline{y} \vee z$
Erase the clause x
Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

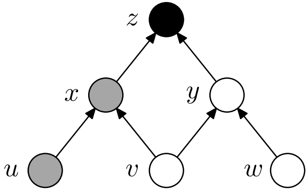


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the clause $\overline{x} \lor \overline{y} \lor z$
Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

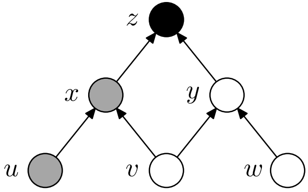


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

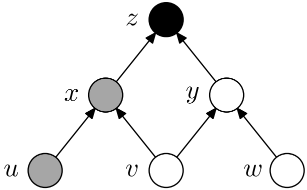


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

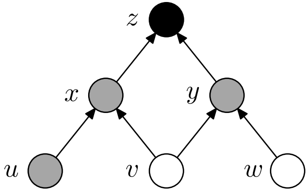


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

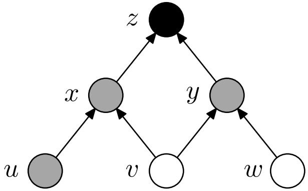


$\overline{v} \lor \overline{w} \lor z$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



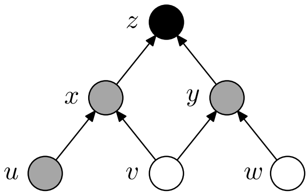
$\overline{v} \lor \overline{w} \lor z$

v

Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$
Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

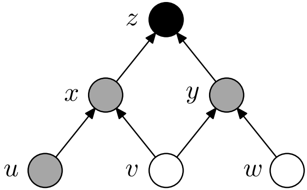


$\overline{v} \lor \overline{w} \lor z$
v
w

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$
Write down axiom 2: v
Write down axiom 3: w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

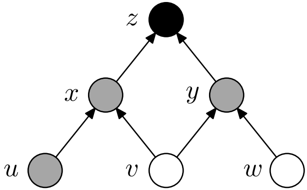


$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$

Erase the clause $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{y} \vee z$
Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

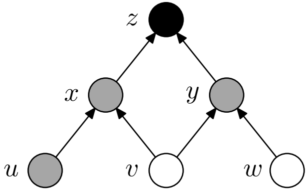


$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

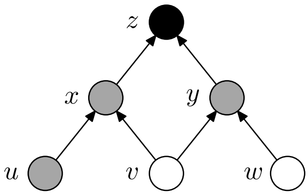


$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

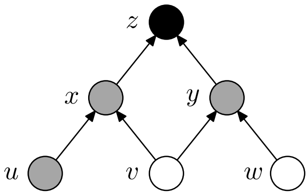


$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



$\overline{v} \lor \overline{w} \lor z$

w

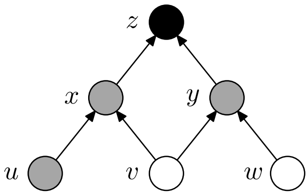
$\overline{z}$

$\overline{w} \lor z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{v} \vee \overline{w} \vee z$

w

$\overline{z}$

$\overline{w} \vee z$

Write down axiom 7: $\overline{z}$

Infer $\overline{w} \vee z$ from

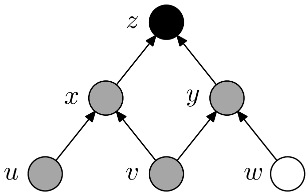
v and $\overline{v} \vee \overline{w} \vee z$

Erase the clause v

Erase the clause $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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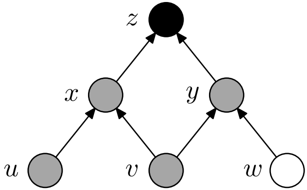
w
$\overline{z}$
$\overline{w} \vee z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$
Erase the clause v
Erase the clause $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



w

$\overline{z}$

$\overline{w} \lor z$

v and $\overline{v} \lor \overline{w} \lor z$

Erase the clause v

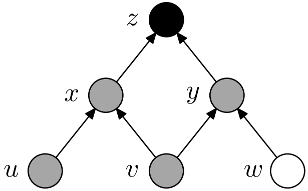
Erase the clause $\overline{v} \lor \overline{w} \lor z$

Infer z from

w and $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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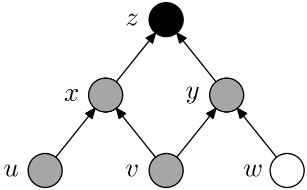
w
$\overline{z}$
$\overline{w} \lor z$
z

v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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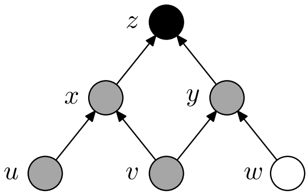
w
$\overline{z}$
$\overline{w} \lor z$
z

Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the clause w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



$\overline{z}$

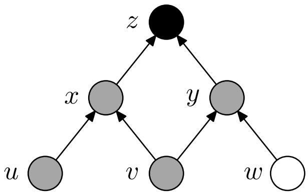
$\overline{w} \lor z$

z

Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the clause w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{z}$

$\overline{w} \vee z$

z

Erase the clause $\overline{v} \vee \overline{w} \vee z$

Infer z from

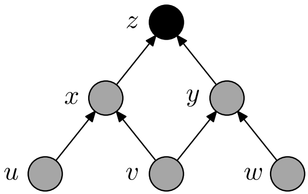
w and $\overline{w} \vee z$

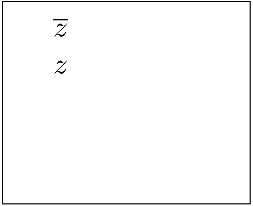
Erase the clause w

Erase the clause $\overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

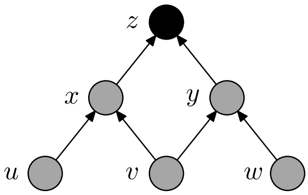




Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the clause w
Erase the clause $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



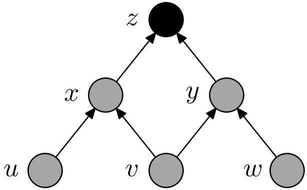
$\overline{z}$

z

w and $\overline{w} \vee z$
Erase the clause w
Erase the clause $\overline{w} \vee z$
Infer 0 from
$\overline{z}$ and z

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



$\overline{z}$

z

0

w and $\overline{w} \lor z$

Erase the clause w

Erase the clause $\overline{w} \lor z$

Infer 0 from

$\overline{z}$ and z

Formal Refutation-Pebbling Correspondence

Theorem (Ben-Sasson '02)

Any refutation translates into black-white pebbling with

- *# moves $\leq$ refutation length*
- *# pebbles $\leq$ # variables on blackboard*

Observation (Ben-Sasson et al. '00)

Any black-pebbles-only pebbling translates into refutation with

- *refutation length $\leq$ # moves*
- *total space $\leq$ # pebbles*

Unfortunately pebbling contradictions are extremely easy w.r.t. clause space! — not what we want

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Key Idea: Variable Substitution

Make formula harder by substituting exclusive or $x_1 \oplus x_2$ of two new variables x_1 and x_2 for every variable x

$$\overline{x} \vee y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$$

$$\Downarrow$$

$$\begin{aligned} & (x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \\ \wedge\ & (x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee y_1 \vee y_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2) \end{aligned}$$

Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for refuting $F[\oplus]$: mimic refutation of F

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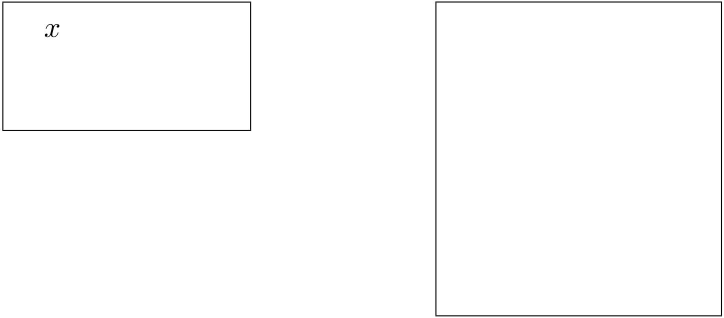

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	$\overline{x}_1 \vee \overline{x}_2$
$\overline{x} \vee y$	$x_1 \vee \overline{x}_2 \vee y_1 \vee y_2$
	$x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2$
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For such refutation of $F[\oplus]$:

- length $\geq$ length for F
- clause space $\geq$ # variables on board in proof for F

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Prove that this is (sort of) best one can do for $F[\oplus]$!

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$...	write $\overline{x} \vee y$ on shadow blackboard
For consecutive XOR blackboard configurations...	can get between corresponding shadow blackboards by legal derivation steps
... (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation ...
... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

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Putting the Pieces Together

Making variable substitutions in pebbling formulas

- lifts lower bound from number of variables to clause space
- maintains upper bound in terms of total space and length

Get our results by

- using known pebbling results from literature of 70s and 80s
- proving a couple of new pebbling results [Nordström '10]
- to get tight trade-offs, showing that resolution proofs can sometimes do better than black-only pebblings [Nordström '10]

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Lower Bounds on Total Space?

Open Question

Are there polynomial-size k-CNF formulas with total refutation space $\Omega\big((\text{size of } F)^2\big)$?

Answer conjectured to be "yes" by [Alekhnovich et al. 2000]

Or can we at least prove a superlinear lower bound measured in # variables?

Trade-offs for Stronger Proof Systems?

Recall key technical theorem: amplify space lower bounds through variable substitution

Almost completely oblivious to which proof system is being studied

Extended to strictly stronger k-DNF resolution proof systems — maybe can be made to work for other stronger systems as well?

Open Question

Can the Substitution Theorem be proven for, say, cutting planes or polynomial calculus, thus yielding time-space trade-offs for these proof systems as well?

Approach in previous works provably will not work, but there are other (related but different) ideas one could try

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Cutting Planes: Informal Description

- Geometric proof system introduced by [Cook, Coullard & Turán '87]
- Translate clauses to linear inequalities for real variables in $[0, 1]$
- For instance, $x \lor y \lor \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$, i.e., $x + y - z \geq 0$
- Manipulate linear inequalities to derive contradiction $0 \geq 1$

Cutting Planes: Inference Rules

Lines in a cutting planes (CP) refutation are linear inequalities with integer coefficients.

Derivation rules:

Variable axioms $\frac{}{x \geq 0}$ and $\frac{}{-x \geq -1}$ for all variables x

Addition $$\frac{\sum a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

Multiplication $$\frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$$ for a positive integer c

Division $$\frac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$$ for a positive integer c

A CP-refutation ends when the inequality $0 \geq 1$ has been derived

Cutting Planes Measures

Length
derivation steps

Line space
Linear inequalities in any configuration
(Analogue of clause space)

Total space
Total # variables in configuration counted with repetitions
+ log of coefficients

Polynomial Calculus

- Algrebraic system introduced by [Clegg, Edmonds & Impagliazzo '96] under the name of "Gröbner proof system"
- Clauses are interpreted as multilinear polynomial equations
- For instance, clause $x \lor y \lor \overline{z}$ gets translated to $xy(1 - z) = 0$ or $xy - xyz = 0$
- Derive contradiction by showing that there is no common root for the polynomial equations corresponding to all the clauses

Polynomial Calculus: Inference Rules

Lines in a polynomial calculus (PC) refutation are multivariate polynomial equations $p = 0$, where $p \in \mathbb{F}[x, y, z, \ldots]$ for some (fixed) field $\mathbb{F}$, typically finite

Customary to omit "$= 0$" and only write p

The derivation rules are as follows, where $\alpha, \beta \in \mathbb{F}$, $p, q \in \mathbb{F}[x, y, z, \ldots]$, and x is any variable:

Boolean axioms $\dfrac{}{x^2 - x}$ for all x (forcing 0/1-solutions)

Linear combination $\dfrac{p \qquad q}{\alpha p + \beta q}$

Multiplication $\dfrac{p}{xp}$

A PC-refutation ends when 1 has been derived (i.e., $1 = 0$)

(Note that multilinearity follows w.l.o.g. from $x^2 = x$)

Polynomial Calculus: Alternate View

Can also (equivalently) consider a PC-refutation to be a calculation in the ideal generated by polynomials corresponding to clauses

Then a refutation concludes by proving that 1 is in this ideal, i.e., that the ideal is everything

Clearly implies that there is no common root

Less obvious: if no common root, then 1 is always in the ideal (requires some algebra)

Polynomial Calculus Measures

Size
Total # monomials in the refutation counted with repetitions

Length
derivation steps
(≈ # polynomial equations counted with repetitions)

(Monomial) space
Maximal # monomials in any configuration counted with repetitions (again an analogue of clause space)

Total space
Total # variables in any configuration counted with repetitions

State-of-the-art for CP and PC

- Strong lower bounds on proof size/length (but only for one formula family in cutting planes)
- But space very poorly understood, if at all
- Nothing known about time-space trade-offs
- CP and PC interesting proof systems, since one could conceivably base strong(er) SAT solvers on them (and also for other reasons)

Some Related Recent Developments

Theorem (Huynh & Nordström, Oct '11)

There are k-CNF formulas refutable in resolution in length $\mathcal{O}(n)$ such that any

- *polynomial calculus refutation in length L and monomial space s has*

$$s \log L \gtrsim \sqrt[4]{n}$$

- *cutting planes refutation in length L and line space s has*

$$s \log L \gtrsim \sqrt[4]{n}$$

Doesn't use substitution theorem, but lifting + communication complexity à la [Beame, Huynh & Pitassi '10]

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Some Even more Recent Developments

Theorem (Filmus, Lauria, Nordström, Thapen, & Zewi, Nov '11)

There are k-CNF formulas that require (almost) linear monomial space in polynomial calculus (and any k-CNF formula can be refuted in linear space).

More Open Problems

- Many other open (theoretical) questions about space in proof complexity
- See recent survey *Pebble Games, Proof Complexity, and Time-Space Trade-offs* at my webpage for details
- To conclude, want to focus on main applied question

Is the Theoretical Model Good Enough?

- Research motivated (among other things) by questions regarding applied SAT solving, but results purely theoretical
- On the face of it, the "blackboard model" for resolution looks quite far from what a DPLL SAT solver actually does
- More recent models in e.g. [Buss et al.'08, Pipatsrisawat & Darwiche '09] seem closer to practice (but not as nice to work with)
- Do our results hold in these models as well?
- Preliminary answer: at least for [Buss et al.'08] this seems to be the case

Is Tractability Captured by Space Complexity?

Open Question

Do our trade-off phenomena show up in real life for state-of-the-art SAT solvers run on pebbling contradictions?

That is, does space complexity capture hardness?

Space suggested as hardness measure in [Ansótegui et al.'08]

Preliminary experiments indicate that pebbling formulas are in fact hard for SAT solvers

Note that pebbling formulas are always extremely easy with respect to length, so hardness in practice would be intriguing

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Take-Home Message

- Modern SAT solvers, although based on old and simple DPLL method, can be enormously successful in practice
- Key issue: minimize time and memory, but we show strong time-space trade-offs that should make this impossible
- Many remaining open questions about space in proof complexity
- Main open practical question: is tractability captured by space complexity?
- Main open theoretical questions: what about stronger algebraic or geometric proof systems?

Some Advertising

- Course on proof complexity given at KTH right now—not too late to join
- I am hiring PhD students and postdocs (start date Aug 2012)

Thank you for your attention!