Rediscovering the Joys of Pebbling

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Theory reading group
April 17, 2013

a.k.a. "On the Relative Strength of Pebbling and Resolution"

The Big Picture

- Satisfiability algorithms
  - Dramatic developments last 10-15 years
  - SAT solvers used on regular basis for large-scale real-world problems
  - Best algorithms based on resolution proof system
  - Bottlenecks: time and memory consumption

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  - Bottlenecks: time and memory consumption
- Pebble games
  - Used in 70s-80s to study programming languages, compiler optimization, et cetera
  - No developments whatsoever last 20-25(?) years
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- This talk
  - What can proof complexity say about time vs space?
  - Connections between resolution and pebble games?

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- This talk
  - What can proof complexity say about time vs space?
  - Connections between resolution and pebble games?
  - And after the break: some pebbling results and proofs

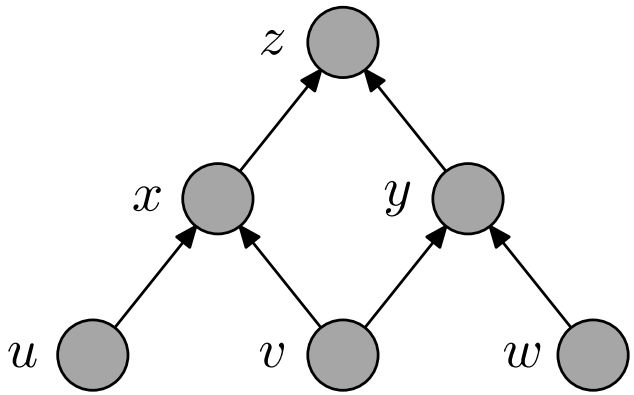
Outline

Just to Check We're on the Same Page...

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$ (assume k fixed)
- Refer to clauses of CNF formula as axioms (as opposed to derived clauses)

Example CNF Formula

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

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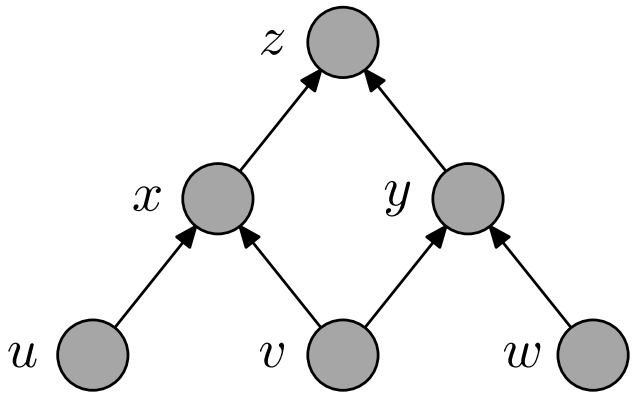

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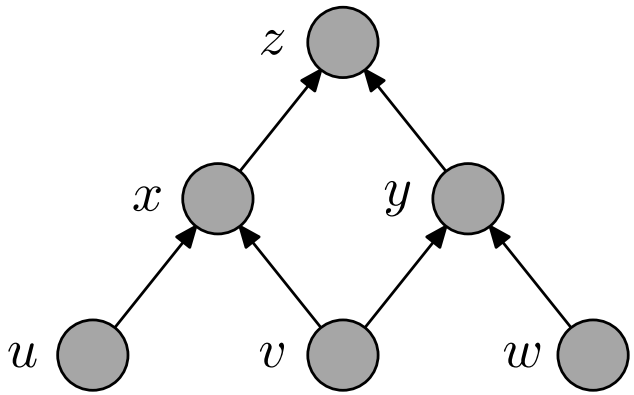

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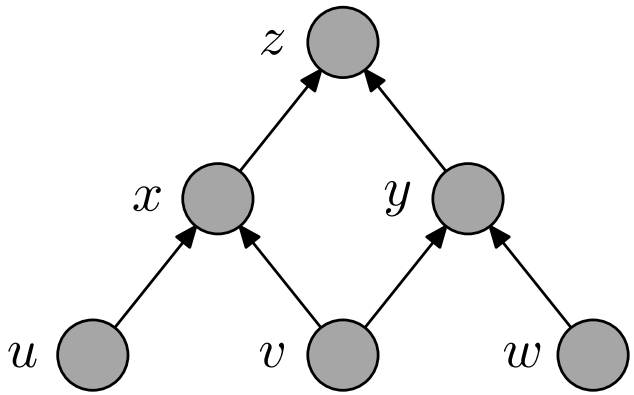

Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

Example Resolution Refutation

1. u
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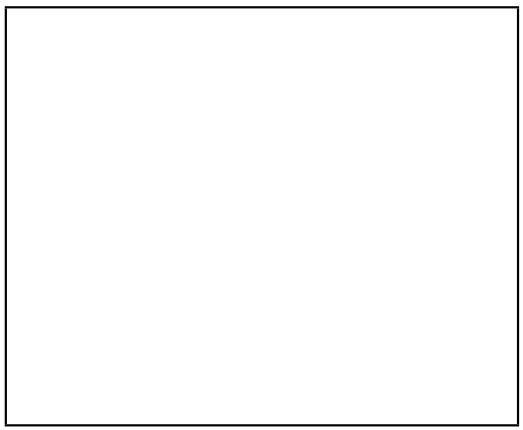


Blackboard bookkeeping	
total # clauses on board	0
max # lines on board	0
max # literals on board	0

Can write down axioms,
erase used clauses or
infer new clauses by resolution rule

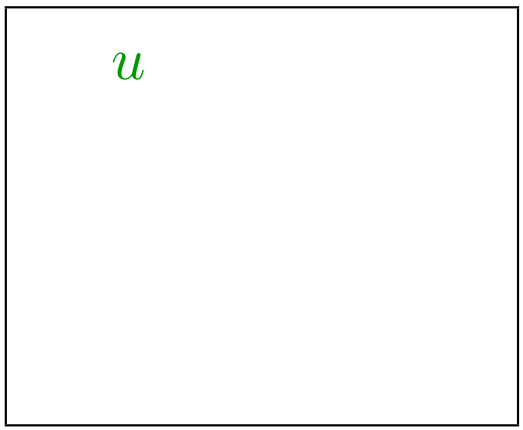
$$\frac{B \vee x \qquad C \vee \overline{x}}{B \vee C}$$

(but only from clauses currently on the board!)

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	1
max # lines on board	1
max # literals on board	1

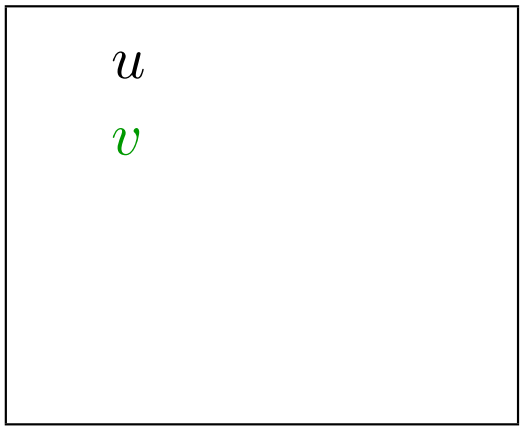


Write down axiom 1: u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	2
max # lines on board	2
max # literals on board	2



Write down axiom 1: u

Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
max # lines on board	3
max # literals on board	5

u

v

$\overline{u} \vee \overline{v} \vee x$

Write down axiom 1: u

Write down axiom 2: v

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
max # lines on board	3
max # literals on board	5

u

v

$\overline{u} \vee \overline{v} \vee x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u

v

$\overline{u} \vee \overline{v} \vee x$

$\overline{v} \vee x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u

v

$\overline{u} \vee \overline{v} \vee x$

$\overline{v} \vee x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$
Erase the line $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u

v

$\overline{v} \vee x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$
Erase the line $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u

v

$\overline{v} \vee x$

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Infer $\overline{v} \vee x$ from

u and $\overline{u} \vee \overline{v} \vee x$

Erase the line $\overline{u} \vee \overline{v} \vee x$

Erase the line u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

v

$\overline{v} \vee x$

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Infer $\overline{v} \vee x$ from

u and $\overline{u} \vee \overline{v} \vee x$

Erase the line $\overline{u} \vee \overline{v} \vee x$

Erase the line u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

v

$\overline{v} \vee x$

u and $\overline{u} \vee \overline{v} \vee x$

Erase the line $\overline{u} \vee \overline{v} \vee x$

Erase the line u

Infer x from

v and $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

v

$\overline{v} \vee x$

x

u and $\overline{u} \vee \overline{v} \vee x$

Erase the line $\overline{u} \vee \overline{v} \vee x$

Erase the line u

Infer x from

v and $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

v

$\overline{v} \vee x$

x

Erase the line $\overline{u} \vee \overline{v} \vee x$

Erase the line u

Infer x from

v and $\overline{v} \vee x$

Erase the line $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
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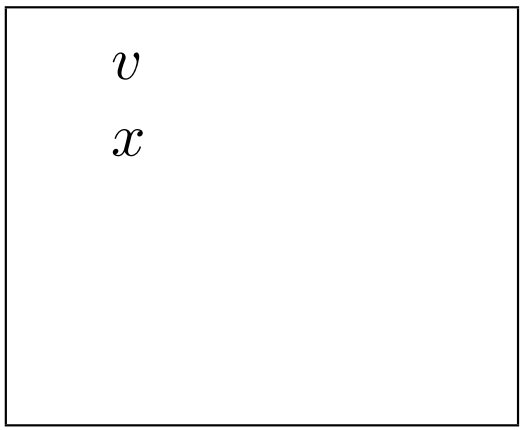
Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

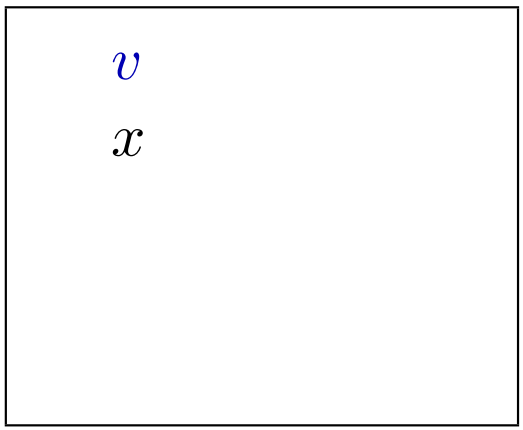


Erase the line $\overline{u} \vee \overline{v} \vee x$
Erase the line u
Infer x from
v and $\overline{v} \vee x$
Erase the line $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

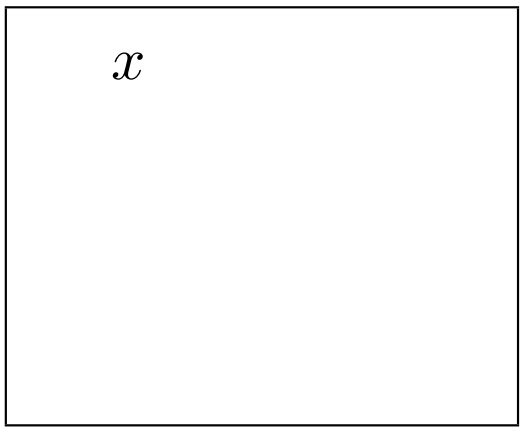


Erase the line u
Infer x from
v and $\overline{v} \vee x$
Erase the line $\overline{v} \vee x$
Erase the line v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7



Erase the line u
Infer x from
v and $\overline{v} \vee x$
Erase the line $\overline{v} \vee x$
Erase the line v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
max # lines on board	4
max # literals on board	7

x

$\overline{x} \vee \overline{y} \vee z$

Infer x from

v and $\overline{v} \vee x$

Erase the line $\overline{v} \vee x$

Erase the line v

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
max # lines on board	4
max # literals on board	7

x

$\overline{x} \vee \overline{y} \vee z$

Erase the line $\overline{v} \vee x$

Erase the line v

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from

x and $\overline{x} \vee \overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x

$\overline{x} \vee \overline{y} \vee z$

$\overline{y} \vee z$

Erase the line $\overline{v} \vee x$

Erase the line v

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from

x and $\overline{x} \vee \overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x

$\overline{x} \vee \overline{y} \vee z$

$\overline{y} \vee z$

Erase the line v

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from

x and $\overline{x} \vee \overline{y} \vee z$

Erase the line $\overline{x} \vee \overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x

$\overline{y} \vee z$

Erase the line v

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from

x and $\overline{x} \vee \overline{y} \vee z$

Erase the line $\overline{x} \vee \overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x

$\overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from

x and $\overline{x} \vee \overline{y} \vee z$

Erase the line $\overline{x} \vee \overline{y} \vee z$

Erase the line x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

$\overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$
Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the line $\overline{x} \vee \overline{y} \vee z$
Erase the line x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
max # lines on board	4
max # literals on board	7

$\overline{y} \vee z$

$\overline{v} \vee \overline{w} \vee y$

Infer $\overline{y} \vee z$ from

x and $\overline{x} \vee \overline{y} \vee z$

Erase the line $\overline{x} \vee \overline{y} \vee z$

Erase the line x

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
max # lines on board	4
max # literals on board	7

$\overline{y} \vee z$

$\overline{v} \vee \overline{w} \vee y$

Erase the line $\overline{x} \vee \overline{y} \vee z$

Erase the line x

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Infer $\overline{v} \vee \overline{w} \vee z$ from

$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$
$\overline{v} \vee \overline{w} \vee z$

Erase the line $\overline{x} \vee \overline{y} \vee z$
Erase the line x
Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \vee z$

$\overline{v} \vee \overline{w} \vee y$

$\overline{v} \vee \overline{w} \vee z$

Erase the line x

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Infer $\overline{v} \vee \overline{w} \vee z$ from

$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Erase the line $\overline{v} \vee \overline{w} \vee y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \vee z$

$\overline{v} \vee \overline{w} \vee z$

Erase the line x

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Infer $\overline{v} \vee \overline{w} \vee z$ from

$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Erase the line $\overline{v} \vee \overline{w} \vee y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \vee z$

$\overline{v} \vee \overline{w} \vee z$

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Infer $\overline{v} \vee \overline{w} \vee z$ from

$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

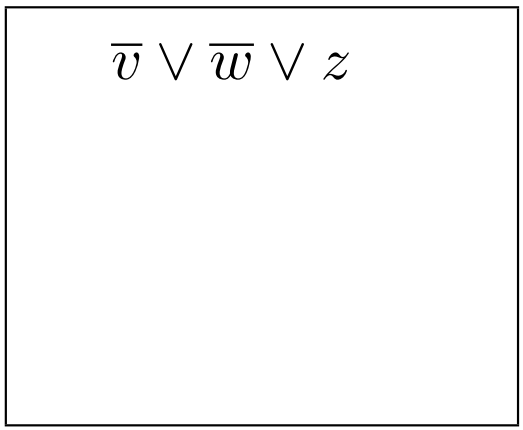
Erase the line $\overline{v} \vee \overline{w} \vee y$

Erase the line $\overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8



Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$
Erase the line $\overline{v} \vee \overline{w} \vee y$
Erase the line $\overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	10
max # lines on board	4
max # literals on board	8

$\overline{v} \vee \overline{w} \vee z$

v

Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$
Erase the line $\overline{v} \vee \overline{w} \vee y$
Erase the line $\overline{y} \vee z$
Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	11
max # lines on board	4
max # literals on board	8

$\overline{v} \vee \overline{w} \vee z$

v

w

$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Erase the line $\overline{v} \vee \overline{w} \vee y$

Erase the line $\overline{y} \vee z$

Write down axiom 2: v

Write down axiom 3: w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
max # lines on board	4
max # literals on board	8

$\overline{v} \vee \overline{w} \vee z$

v

w

$\overline{z}$

Erase the line $\overline{v} \vee \overline{w} \vee y$
Erase the line $\overline{y} \vee z$
Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
max # lines on board	4
max # literals on board	8

$\overline{v} \vee \overline{w} \vee z$

v

w

$\overline{z}$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \vee \overline{w} \vee z$

v

w

$\overline{z}$

$\overline{w} \vee z$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \vee \overline{w} \vee z$

v

w

$\overline{z}$

$\overline{w} \vee z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$
Erase the line v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \vee \overline{w} \vee z$

w

$\overline{z}$

$\overline{w} \vee z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$
Erase the line v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \vee \overline{w} \vee z$

w

$\overline{z}$

$\overline{w} \vee z$

Write down axiom 7: $\overline{z}$

Infer $\overline{w} \vee z$ from

v and $\overline{v} \vee \overline{w} \vee z$

Erase the line v

Erase the line $\overline{v} \vee \overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

w

$\overline{z}$

$\overline{w} \vee z$

Write down axiom 7: $\overline{z}$

Infer $\overline{w} \vee z$ from

v and $\overline{v} \vee \overline{w} \vee z$

Erase the line v

Erase the line $\overline{v} \vee \overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

w

$\overline{z}$

$\overline{w} \vee z$

v and $\overline{v} \vee \overline{w} \vee z$

Erase the line v

Erase the line $\overline{v} \vee \overline{w} \vee z$

Infer z from

w and $\overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

w

$\overline{z}$

$\overline{w} \vee z$

z

v and $\overline{v} \vee \overline{w} \vee z$

Erase the line v

Erase the line $\overline{v} \vee \overline{w} \vee z$

Infer z from

w and $\overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

w
$\overline{z}$
$\overline{w} \vee z$
z

Erase the line v
Erase the line $\overline{v} \vee \overline{w} \vee z$
Infer z from
w and $\overline{w} \vee z$
Erase the line w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$

$\overline{w} \vee z$

z

Erase the line v

Erase the line $\overline{v} \vee \overline{w} \vee z$

Infer z from

w and $\overline{w} \vee z$

Erase the line w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$

$\overline{w} \vee z$

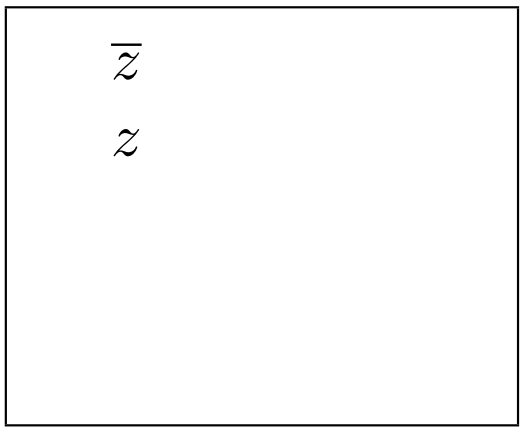
z

Erase the line $\overline{v} \vee \overline{w} \vee z$
Infer z from
w and $\overline{w} \vee z$
Erase the line w
Erase the line $\overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

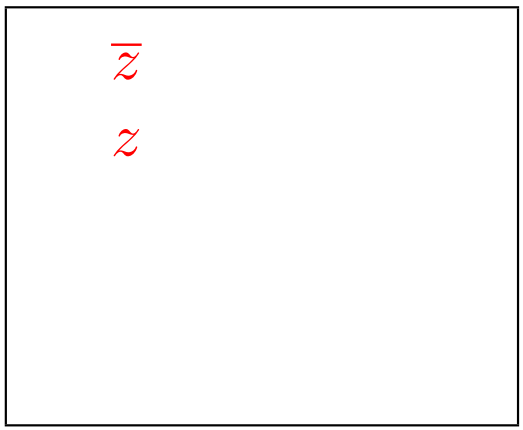


Erase the line $\overline{v} \vee \overline{w} \vee z$
Infer z from
w and $\overline{w} \vee z$
Erase the line w
Erase the line $\overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8



w and $\overline{w} \vee z$
Erase the line w
Erase the line $\overline{w} \vee z$
Infer $\bot$ from
$\overline{z}$ and z

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	15
max # lines on board	5
max # literals on board	8

$\overline{z}$

z

$\bot$

w and $\overline{w} \vee z$

Erase the line w

Erase the line $\overline{w} \vee z$

Infer $\bot$ from

$\overline{z}$ and z

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

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Length

clauses written on blackboard counted with repetitions

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- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

clauses written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

$$x$$
$$\overline{y} \vee z$$
$$\overline{v} \vee \overline{w} \vee y$$

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

clauses written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

1. x
2. $\overline{y} \vee z$
3. $\overline{v} \vee \overline{w} \vee y$

Clause space: 3

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

clauses written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

$$x^1$$
$$\overline{y}^2 \vee\ z^3$$
$$\overline{v}^4 \vee\ \overline{w}^5 \vee\ y^6$$

Clause space: 3

Total space: 6

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

clauses written on blackboard counted with repetitions

(in our example resolution refutation 15)

Space

Somewhat less straightforward — several ways of measuring

$$x$$
$$\overline{y} \vee z$$
$$\overline{v} \vee \overline{w} \vee y$$

Clause space: 3

(in our refutation 5)

Total space: 6

(in our refutation 8)

Length and Space Bounds

Let n = size of formula (# symbols)

Length: at most 2^n
Lower bound $\exp(\Omega(n))$ [Urquhart '87, Chvátal & Szemerédi '88]

Clause space: at most n
Lower bound $\Omega(n)$ [Torán '99, Alekhnovich et al. '00]

Length-Space Trade-offs

Small space $\Rightarrow$ short length

$\exists$ constant clause space refutation $\Rightarrow$ $\exists$ polynomial length refutation [Atserias & Dalmau '03]

Converse **not** true

$\exists$ formulas refutable in linear length requiring $n/\log n$ clause space [Ben-Sasson & Nordström '08]

Dramatic length-space trade-offs in worst case

[Ben-Sasson & Nordström '11] and [Beame, Beck & Impagliazzo '12] showed $\exists$ formulas that

- are refutable in small length
- are refutable in small space
- allow no meaningful simultaneous optimization (minimizing one measure incurs severe penalty in the other)

What We **Don't** Know About Space

Open Question

Total space quadratic in worst case — is this tight? Not even superlinear lower bounds known!

What We **Don't** Know About Space

Open Question

Total space quadratic in worst case — is this tight? Not even superlinear lower bounds known!

Open Question

3-CNF formula refutable in clause space $s \Rightarrow$ length $\mathcal{O}(n^s)$. Can you do space $\mathcal{O}(s)$ and length $n^{\mathcal{O}(s)}$ simultaneously?

We Really Don't Understand Space That Well...

All lower bounds on space seem to follow (with hindsight) from

- bounds for other measures that we understand better (e.g. width),

  or

- connections to pebble games

Pebbling and Time-Space Trade-offs

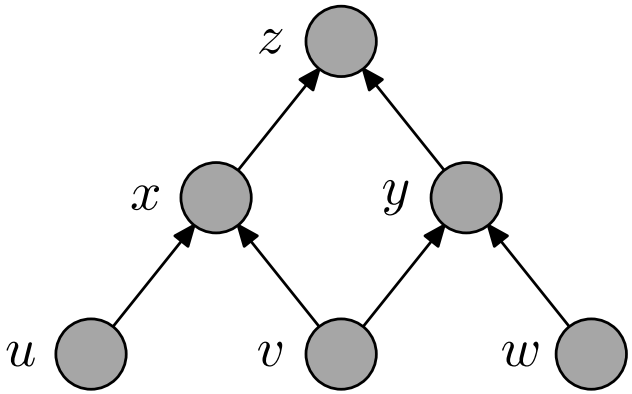
Questions about time-space trade-offs fundamental in TCS

In particular, well-studied (and well-understood) for pebble games modelling calculations described by DAGs ([Cook & Sethi '76] and many others)

- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

The Black-White Pebble Game

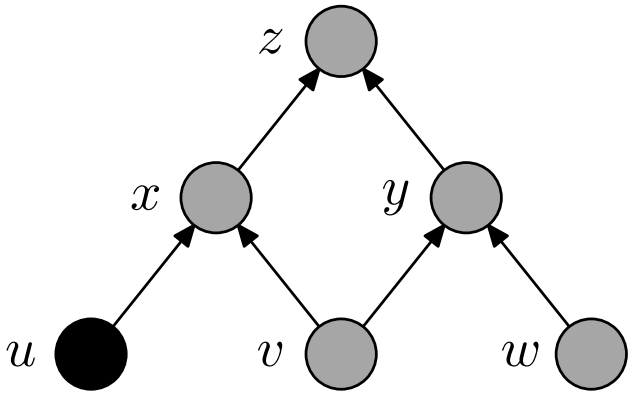
Goal: get single black pebble on sink vertex of G



# moves	0
Current # pebbles	0
Max # pebbles so far	0

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

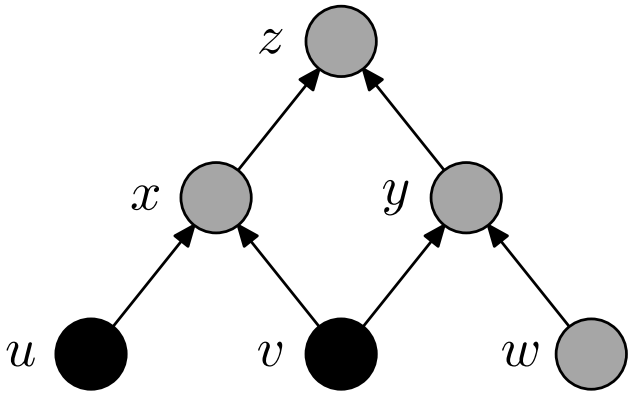


# moves	1
Current # pebbles	1
Max # pebbles so far	1

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

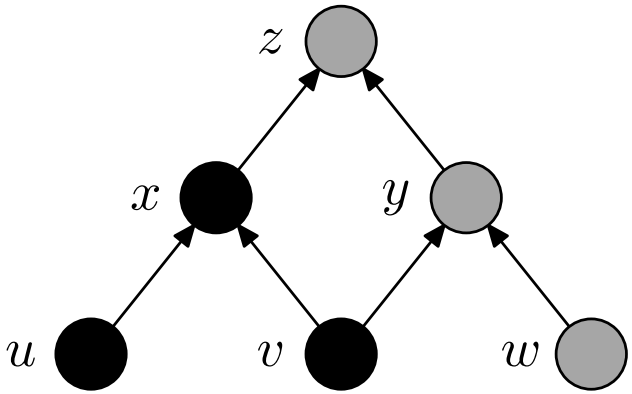


# moves	2
Current # pebbles	2
Max # pebbles so far	2

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

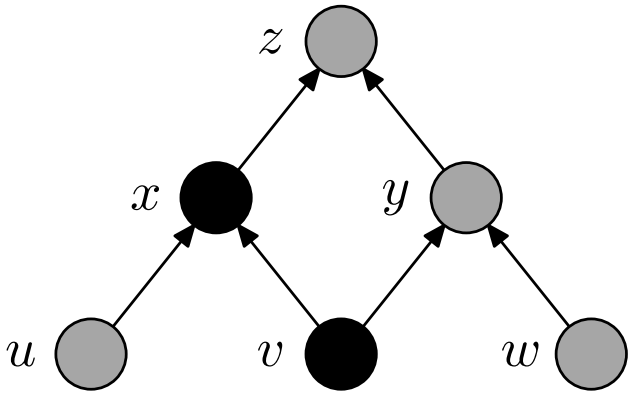


# moves	3
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

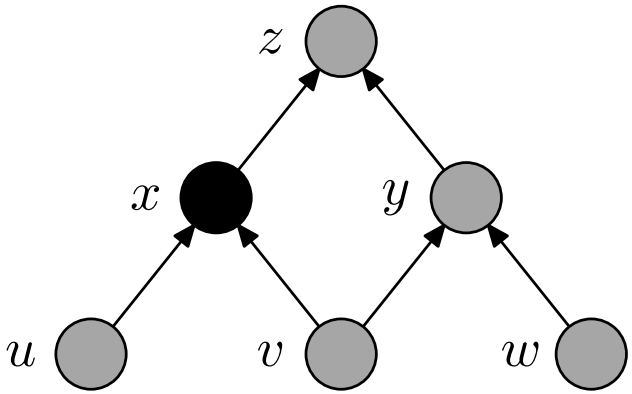


# moves	4
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

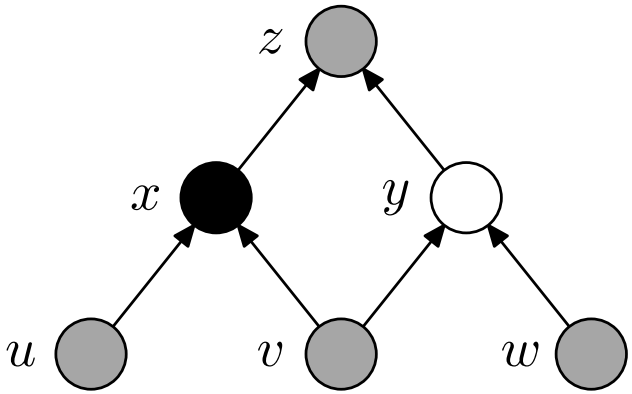


# moves	5
Current # pebbles	1
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

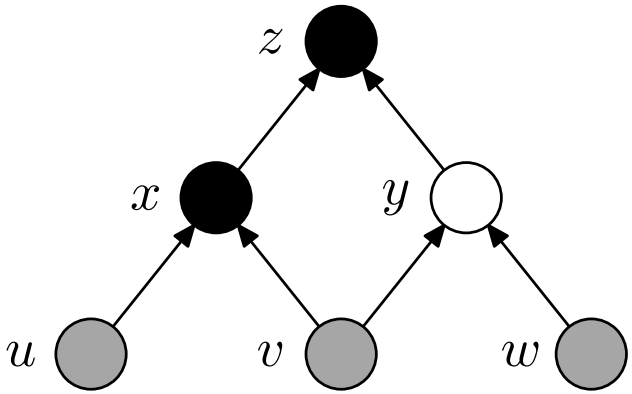


# moves	6
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

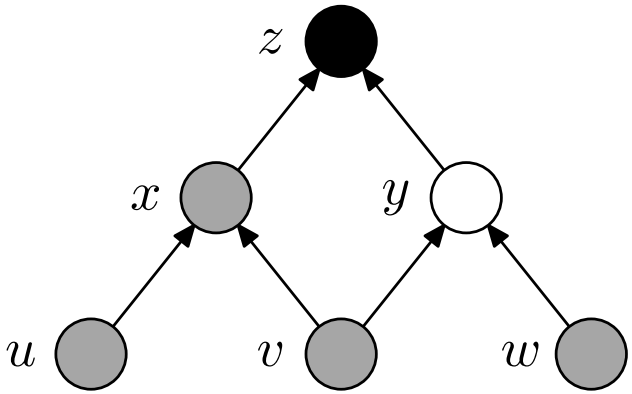


# moves	7
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

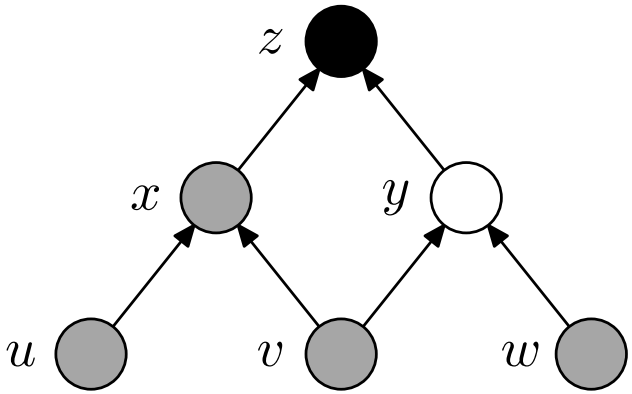


# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G



# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

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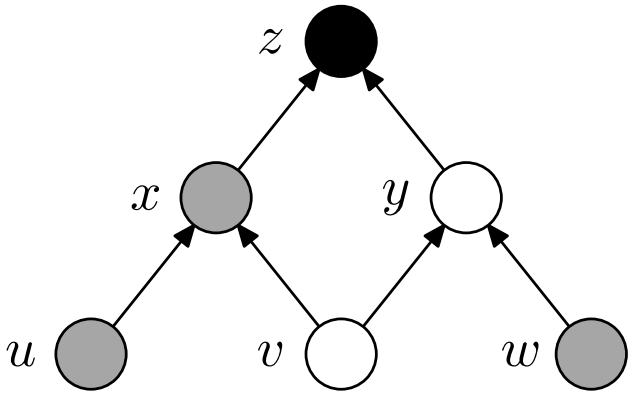

# moves	9
Current # pebbles	3
Max # pebbles so far	3

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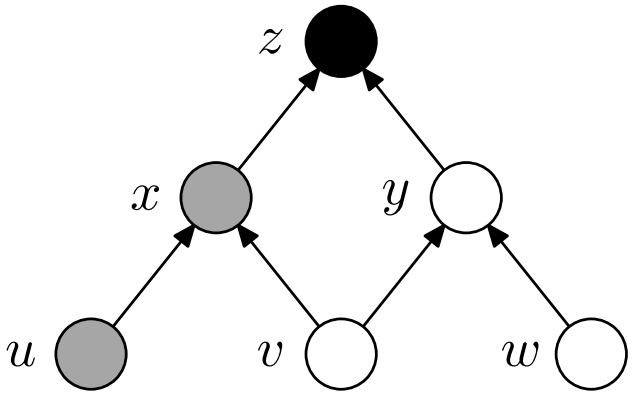

# moves	10
Current # pebbles	4
Max # pebbles so far	4

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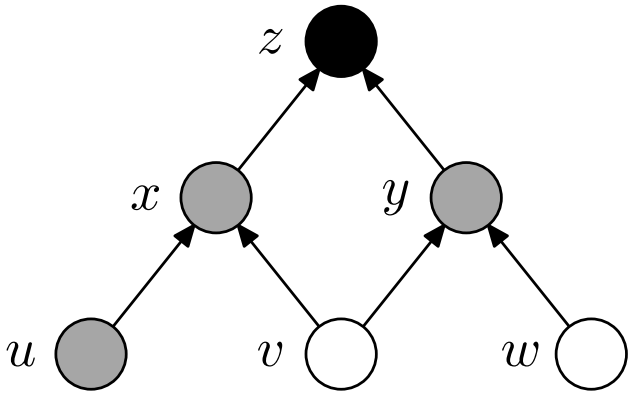

# moves	11
Current # pebbles	3
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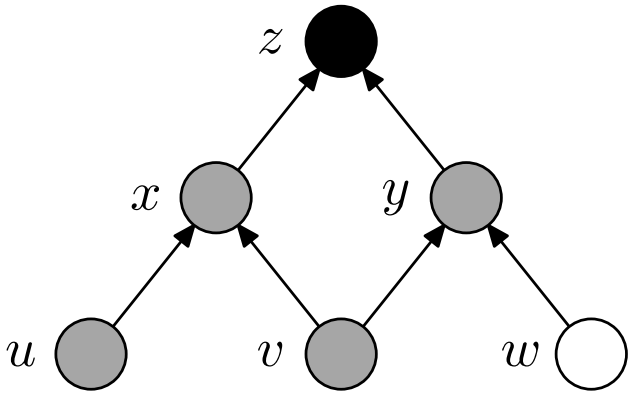

# moves	12
Current # pebbles	2
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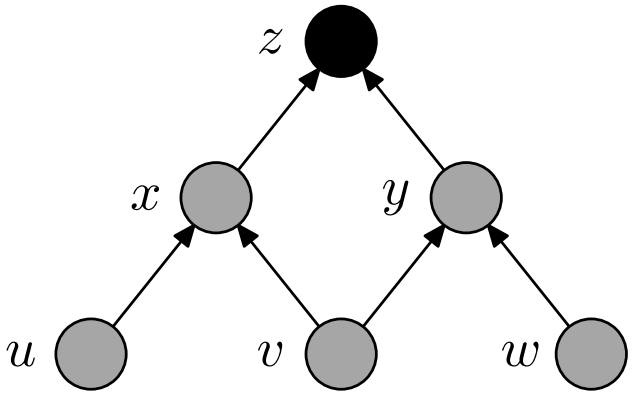

# moves	13
Current # pebbles	1
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
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More About Pebbling

- Black pebbling: Same game but black pebbles only
- Rich literature on both black and black-white pebbling
- Black-white pebbling can save square root over black pebbling space [Wilber '85, Kalyanasundaram & Schnitger '88]*
- But never more [Meyer auf der Heide '81]
- However, transformation from black-white to black-only pebbling incurs exponential time blow-up

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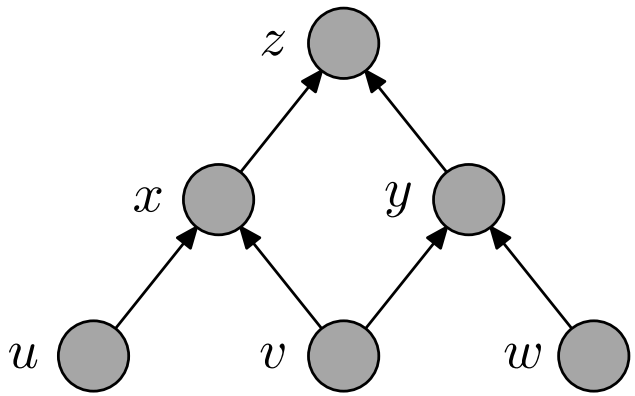
(*) Last two papers in the pebbling literature?

Pebbling Contradictions

CNF formulas encoding pebble game on DAGs

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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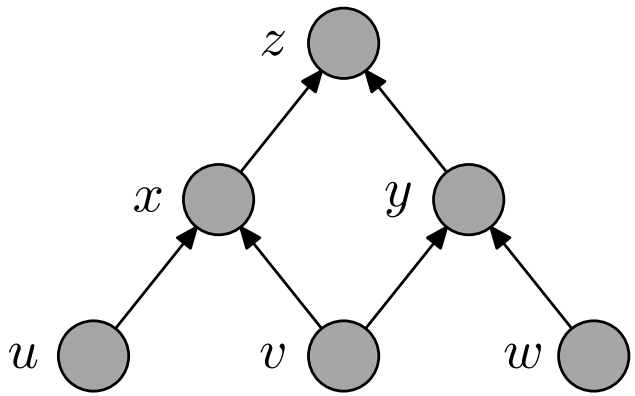

- sources are true
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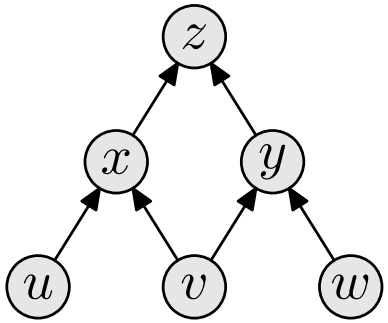
- but sink is false

Studied by [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and others

The Actual* Formulas We Need



$$
\begin{aligned}
&(u_1 \vee u_2) \\
\wedge\ &(v_1 \vee v_2) \\
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\wedge\ &(\overline{u}_1 \vee \overline{v}_1 \vee x_1 \vee x_2) \\
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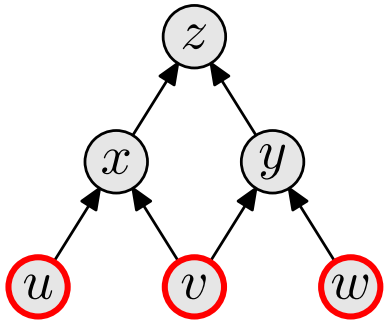

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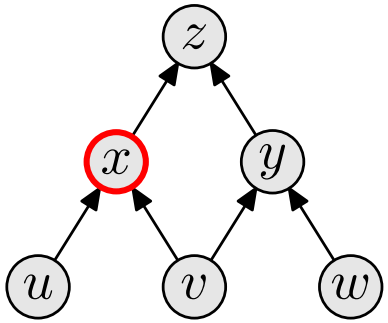

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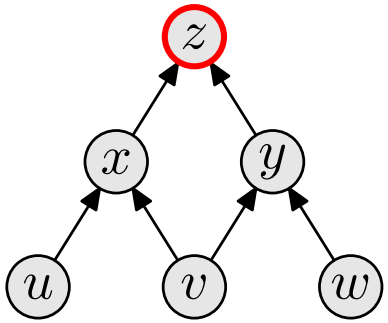

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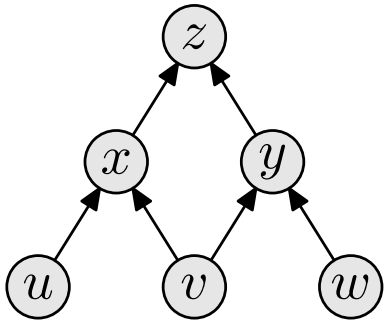

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(*) In fact, they are a bit more involved, but let's stick with this for the purposes of this talk

From Resolution to Black-White Pebbling

Black-white pebbling models non-deterministic computation

- black pebbles $\Leftrightarrow$ computed results
- white pebbles $\Leftrightarrow$ guesses needing to be verified

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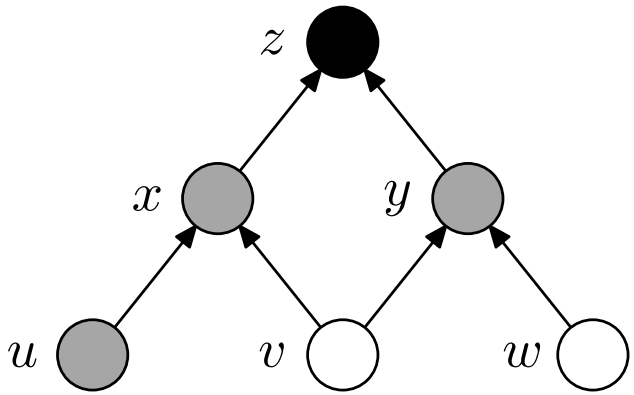

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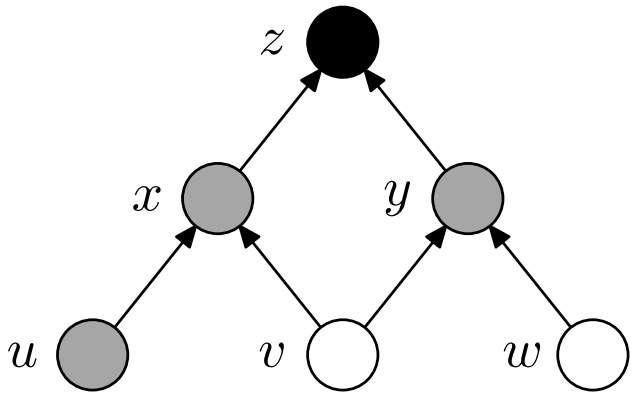

"Know z assuming v, w"

Corresponds to $(v \wedge w) \rightarrow z$, i.e., blackboard clauses

$$\overline{v}_1 \vee \overline{w}_1 \vee z_1 \vee z_2$$
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Formal Refutation-Pebbling Correspondence

Theorem (Ben-Sasson & Nordström '11)

Any refutation translates into black-white pebbling with

- $\#$ *moves* $\leq$ *refutation length*
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- *# moves $\leq$ refutation length*
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Observation (Ben-Sasson et al. '00)

Any black-pebbles-only pebbling translates into refutation with

- *refutation length $\leq$ # moves*
- *total space $\leq$ # pebbles*

Proof: Just derive $v_1 \vee v_2$ inductively when vertex v is pebbled.

A Fatal Gap and How to Close It

There is a gap in the reductions!

- From resolution to black-white pebbling
- From pebbling to resolution only for black pebbling
- Why worry — lose only square root? No, everything! (Due to exponential time blow-up)

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What to do?

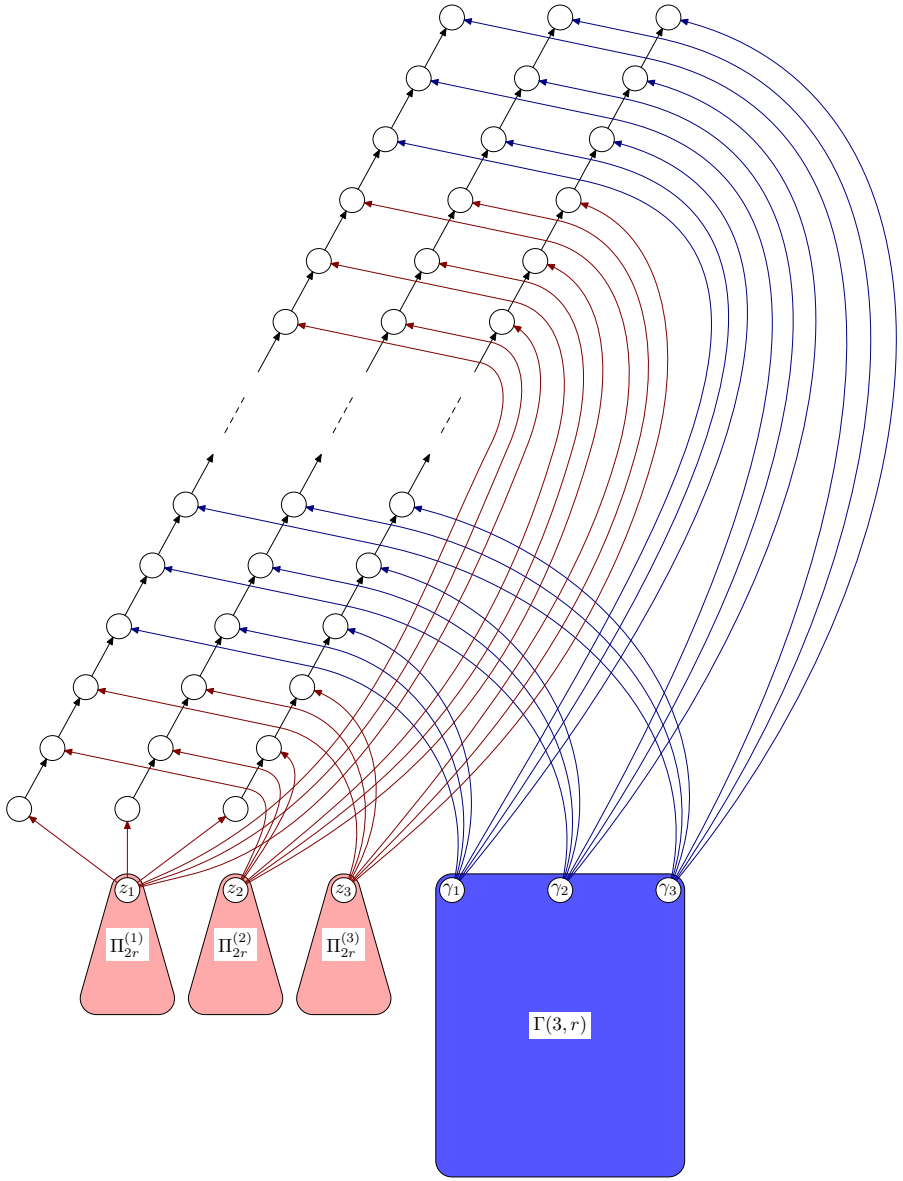
1. Find graphs with (essentially) same trade-off properties for black-white and black-only pebbling
2. Improve reductions between resolution and pebbling

We contribute in both directions

A Picture Says More Than a Thousand Words...

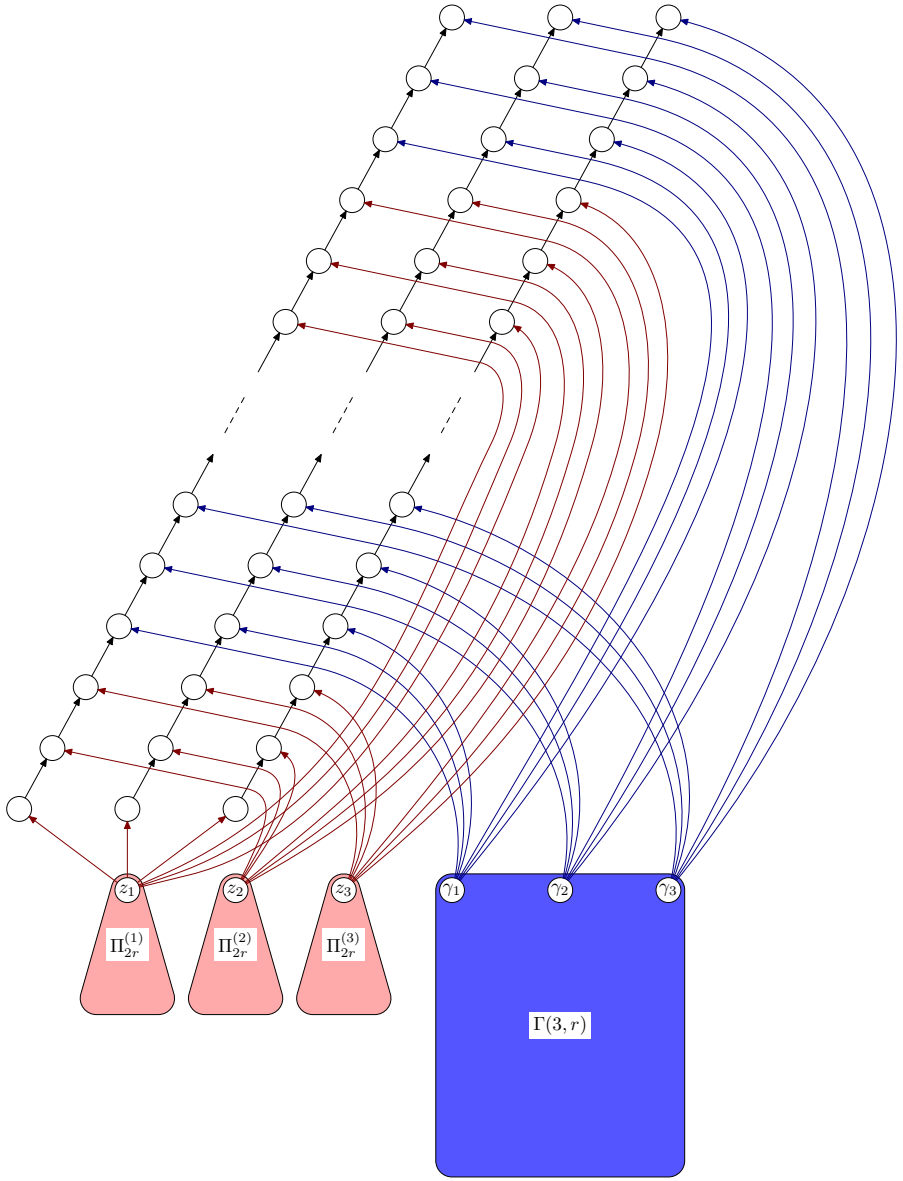
A couple of words about the pebbling result nevertheless:

- Take graph family from [Carlson & Savage '80]
- Black pebbling bounds known (upper and lower)
- Tweak graphs slightly...
- And prove matching black-white lower bounds



Outlining the Argument

Slightly more detailed proof sketch:



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Slightly more detailed proof sketch:

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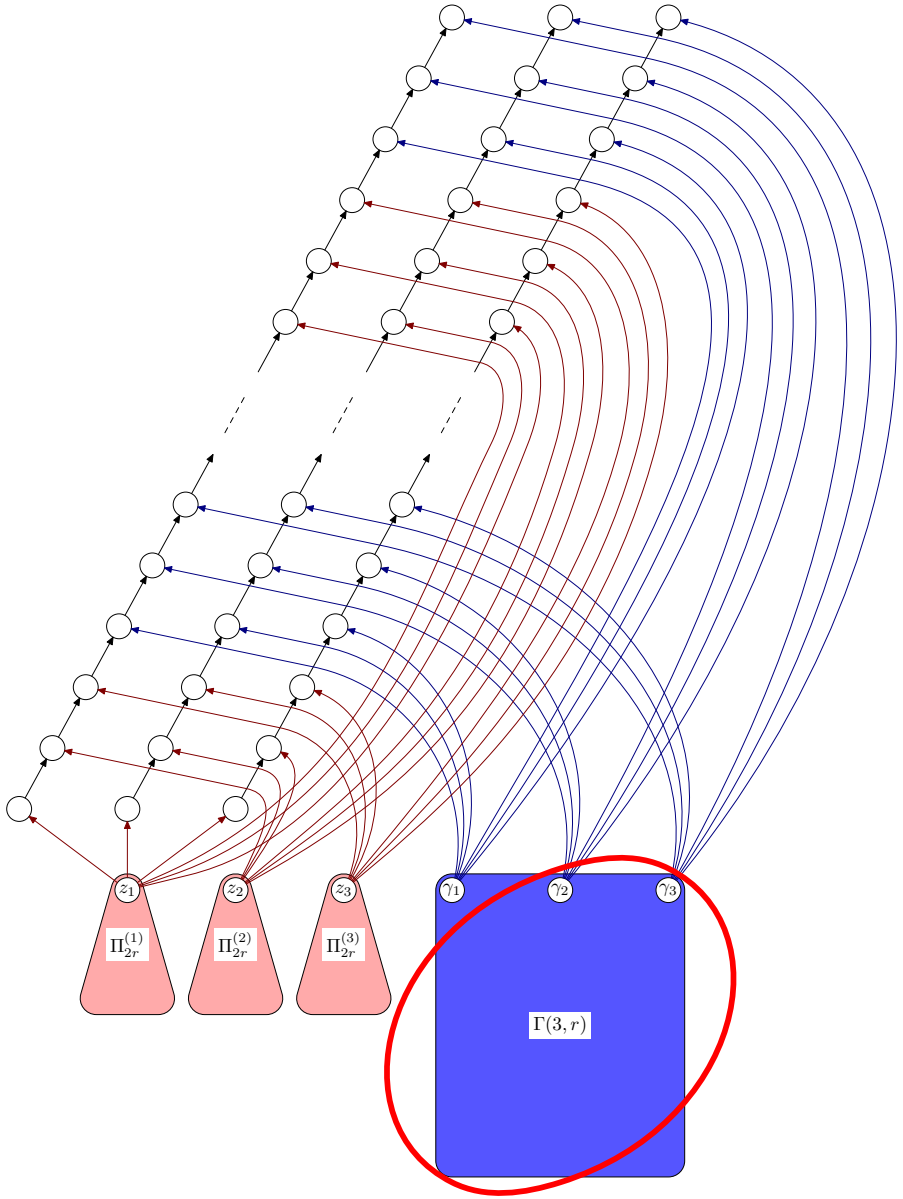

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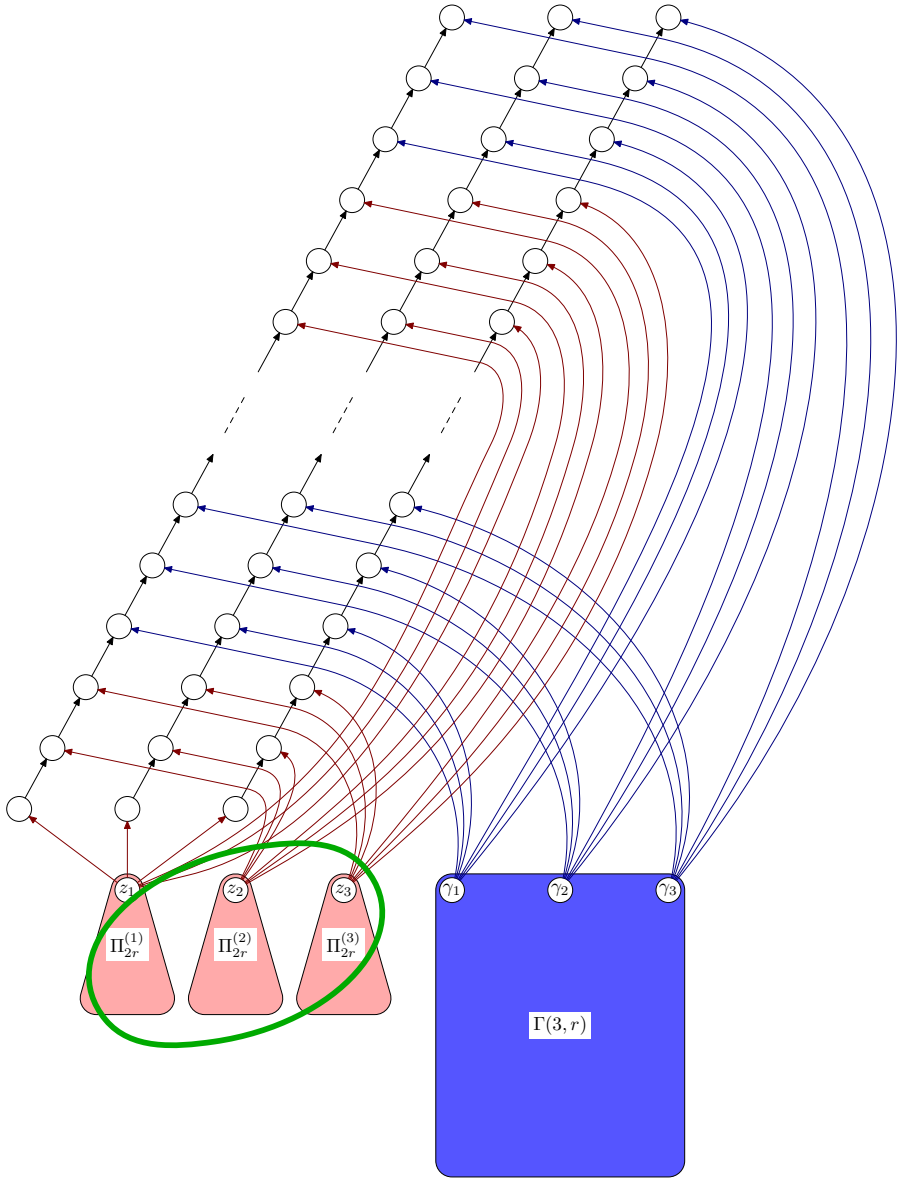

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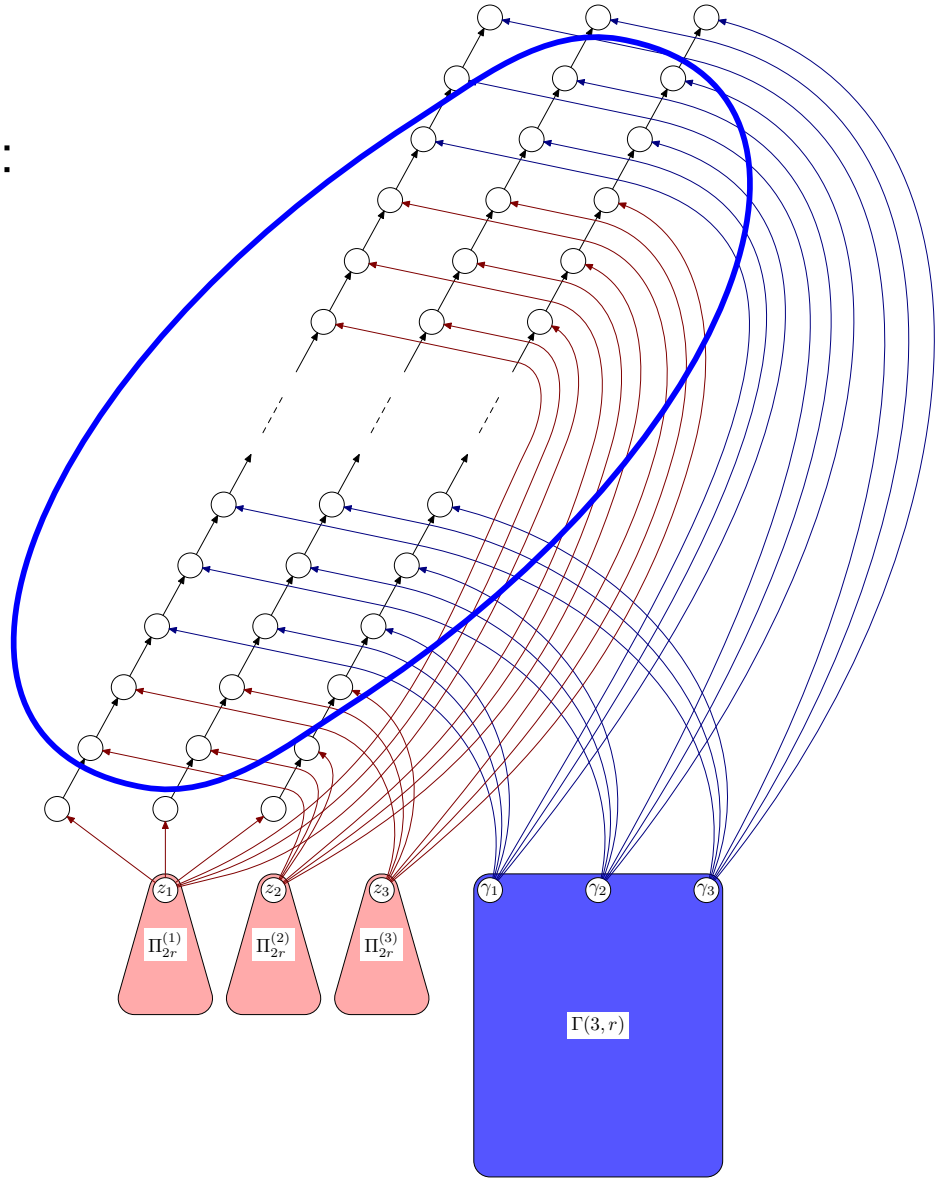

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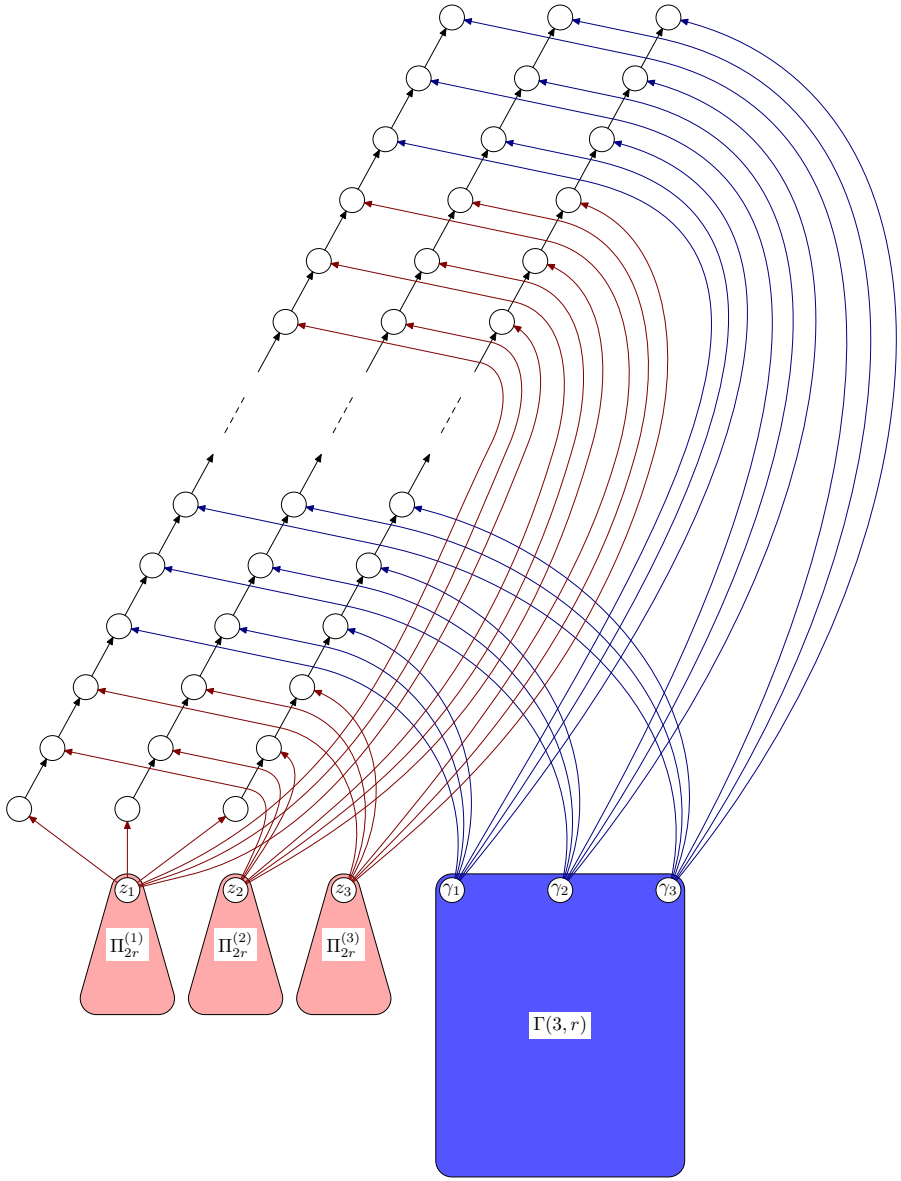
Return to this after the break...



A Naive Idea for Simulating Black-White Pebbling

Run the intuition from [Ben-Sasson & Nordström '11] in reverse

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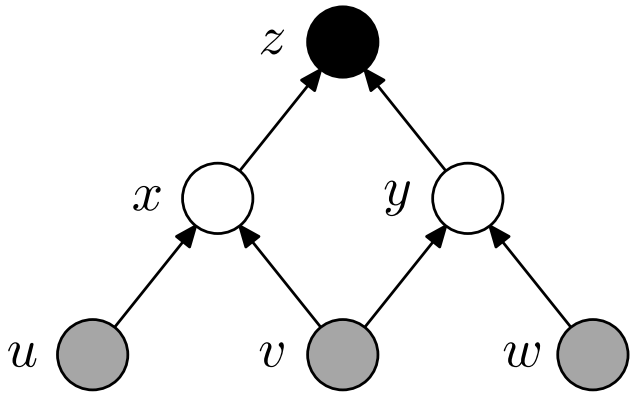

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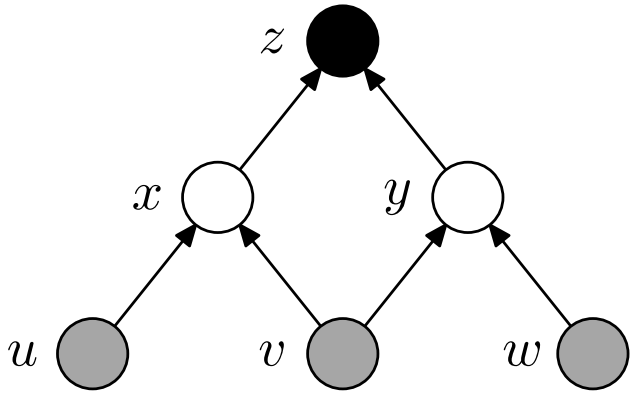

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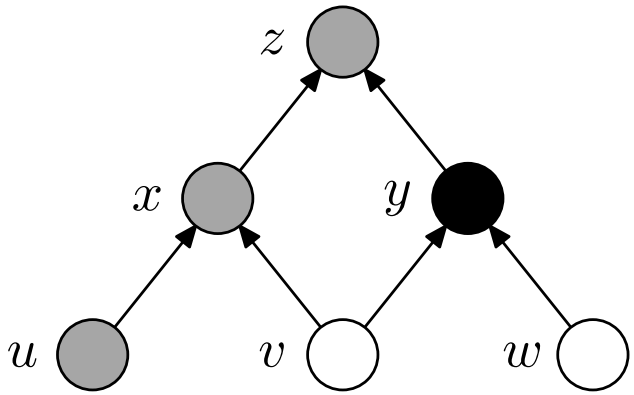

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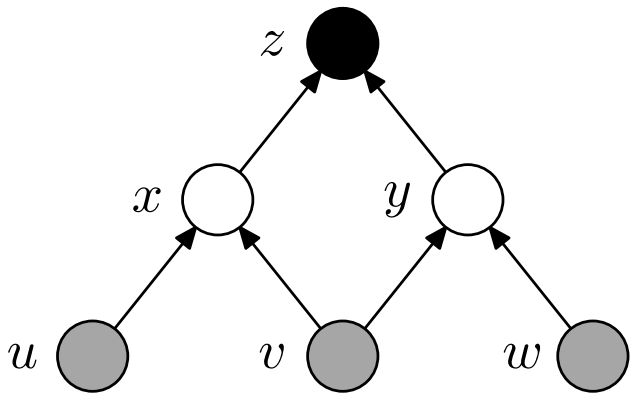
What happens when we try to simulate a pebbling that "combines" these two configurations?



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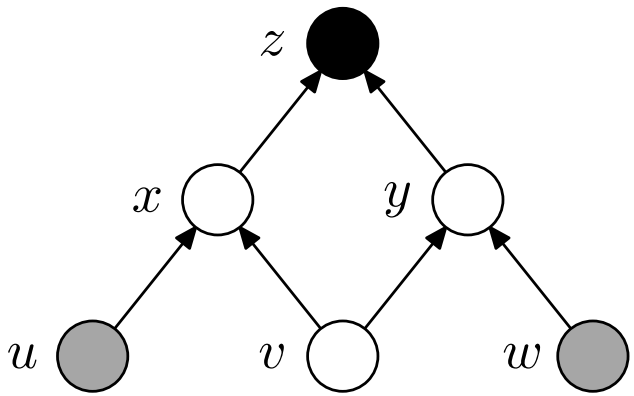

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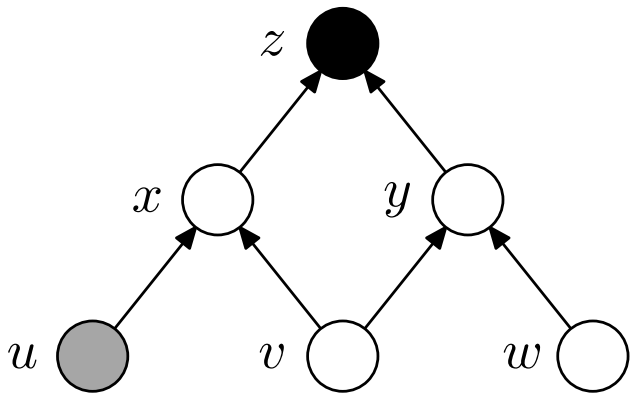

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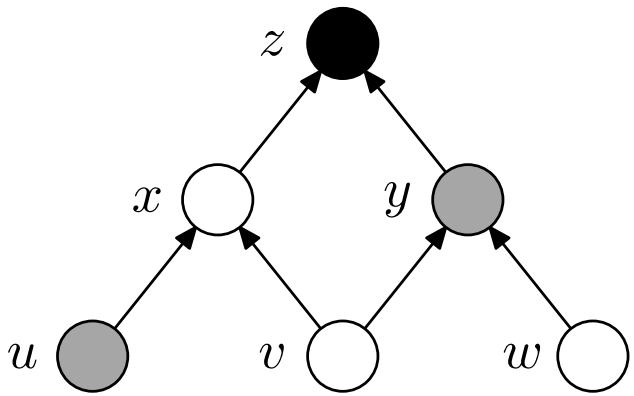

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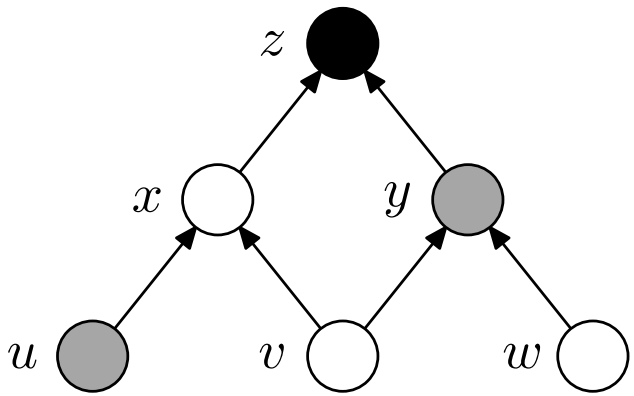

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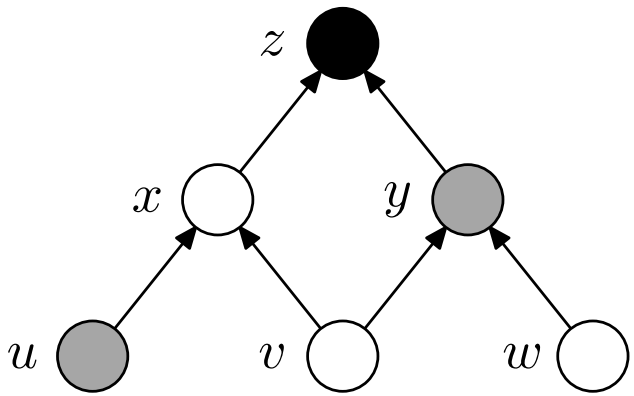

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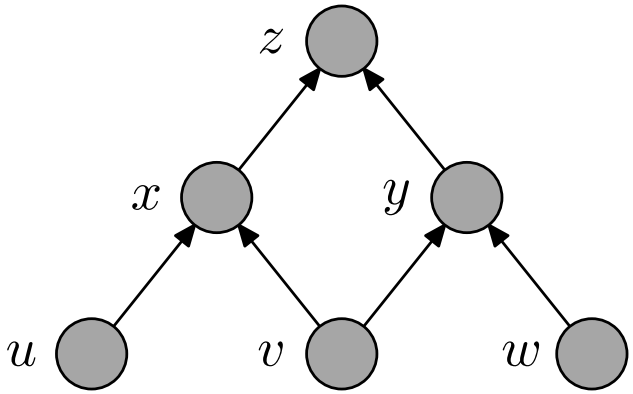
Went only from 2 to 3 white pebbles, but # clauses doubled

Exponential blow-up for naive simulation in worst case

Measure Nondeterminism More Finely

Keep track of for each black pebble which white pebbles it depends on



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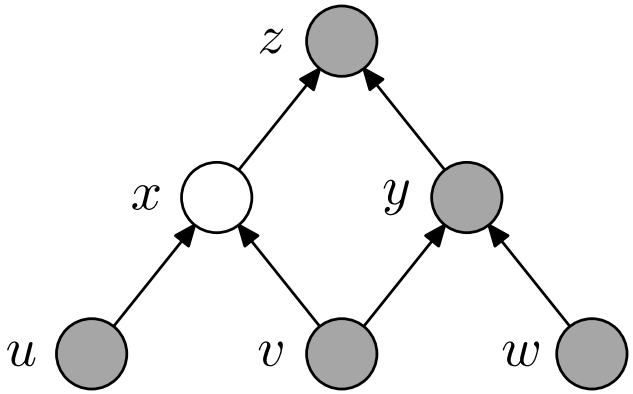

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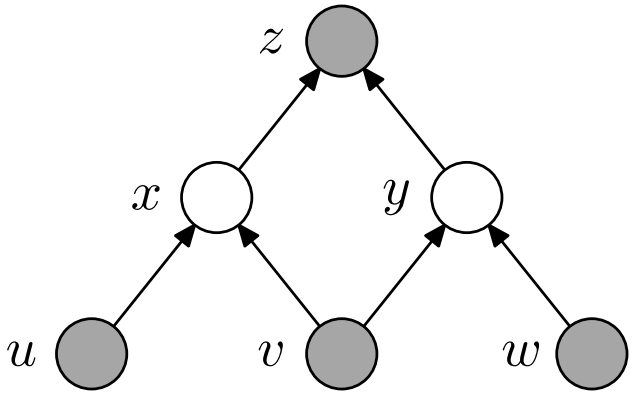

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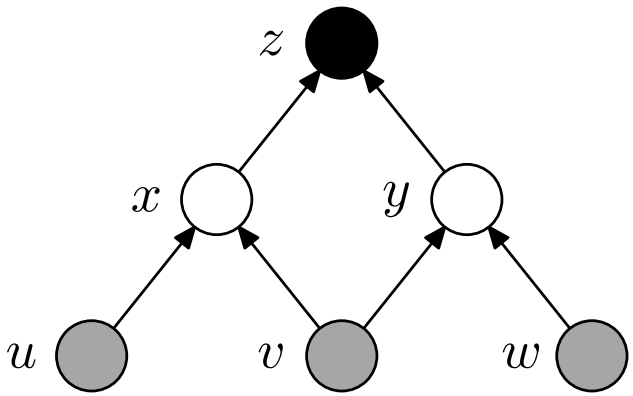

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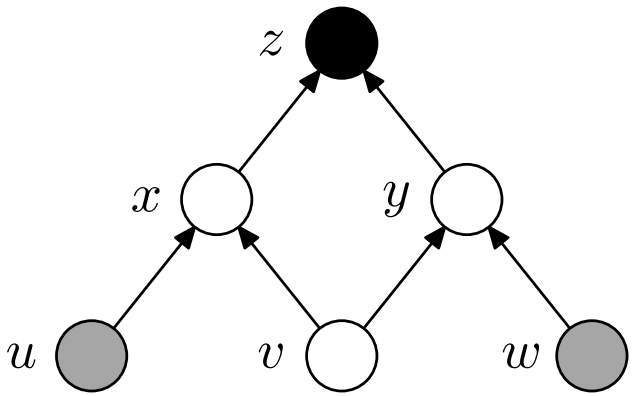

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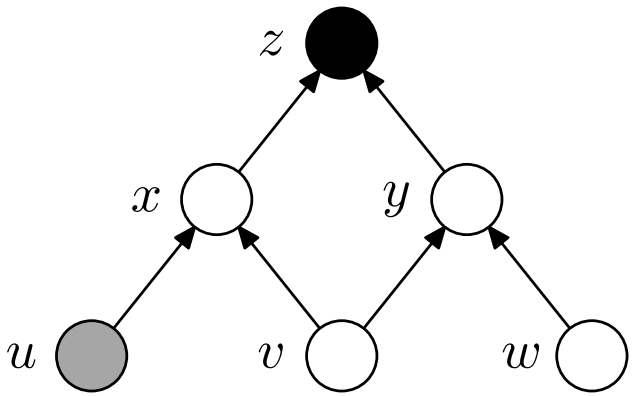

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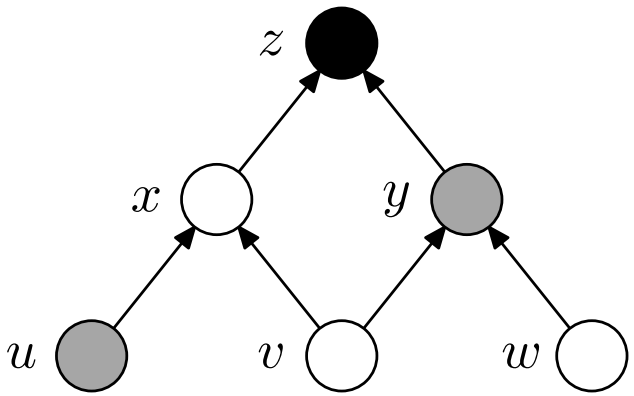

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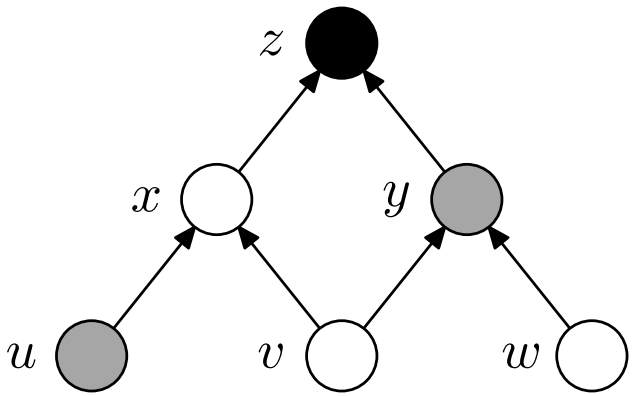

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Black on z dependent on whites on $\{x, y\}$
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Require that each black pebble depend on at most $\mathcal{O}(1)$ white pebbles

Black-white pebbling with "limited nondeterminism"

What Does This Buy Us?

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- Turns out all known pebbling separation results for black-white vs. black pebbling can be matched by pebblings with limited nondeterminism
- Yields tight space bounds and time-space trade-offs for pebbling formulas over such graphs
- So, in particular, not possible to reduce from resolution to black-only pebbling

Resolution and Pebbling

Can we reduce from general black-white pebbling to resolution?

Open Question 1

Can resolution on pebbling formulas always simulate black-white pebbling?

Might or might not be true...

Pebbling with Limited Nondeterminism

Open Question 2

Can pebbling with limited nondeterminism always simulate black-white pebbling?

Pebbling with Limited Nondeterminism

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Can pebbling with limited nondeterminism always simulate black-white pebbling?

Affirmative answer to Question 2 would immediately answer Question 1 as well

Would seem a bit surprising, though...

Candidate for refuting Question 2: Graphs in [Wilber '85]

Space in Resolution

Open Question 3

Total space quadratic in worst case — is this tight? Not even superlinear lower bounds known!

Open Question 4

3-CNF formula refutable in clause space $s \Rightarrow$ length $\mathcal{O}(n^s)$. Can you do space $\mathcal{O}(s)$ and length $n^{\mathcal{O}(s)}$ simultaneously?

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For space $s = 3$ (minimum): No, space-3 refutation can always be carried out in linear length

But space $s = 4$ already wide open...

Take-Home Message

- There are strong (and surprising!) connections between resolution and pebble games
- But still not fully clarified — how tight reductions can we get?
- Also proof space not well-understood — many (simple) remaining open questions
- See survey *Pebble Games, Proof Complexity, and Time-Space Trade-offs* on my webpage for details

Thank you for your attention!