Narrow Proofs May Be Spacious: Separating Space and Width in Resolution

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Executive Summary of Talk (1 / 2)

Resolution: proof system for refuting CNF formulas
Perhaps *the* most studied system in proof complexity
Also used in many real-world automated theorem provers

- Haken (1985): exponential lower bound on proof length (# clauses in a resolution proof)
- Ben-Sasson & Wigderson (1999): lower bound on length in terms of proof width (size of largest clause in proof)
- Results on width lead to question whether other complexity measures could yield interesting insights as well

Executive Summary of Talk (2 / 2)

- Esteban & Torán (1999): proof space
  (maximal # clauses in memory while verifying proof)
- Many lower bounds for space proven
  All turned out to match width bounds!
  Coincidence?
- Atserias & Dalmau (2003): space $\geq$ width $-$ constant for k-CNF formulas
- Problem left open: Do space and width coincide or not?

We resolve this question: separation of space and width

Outline

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \ldots \vee a_k$: set of literals
  At most k literals: k-clause
- CNF formula $F = C_1 \wedge \ldots \wedge C_m$: set of clauses
  k-CNF formula: CNF formula consisting of k-clauses
  (assume k fixed)

Some More Notation and Terminology

- *Vars*$(\cdot)$: set of variables in clause or formula
  Lit$(\cdot)$: set of literals in clause or formula
- Truth value assignment α makes
  - clause true if one literal true
  - CNF formula true if all clauses true
- $F \vDash D$: semantical implication, $\alpha(F)$ true $\Rightarrow \alpha(D)$ true for all truth value assignments α
- $[n] = \{1, 2, \ldots, n\}$

Resolution Rule

Resolution rule:

$$\frac{B \lor x \quad C \lor \overline{x}}{B \lor C}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $C_1, C_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

Prove F unsatisfiable by deriving the unsatisfiable empty clause 0 (the clause with no literals) from F by resolution

Resolution Rule

Resolution rule:

$$\frac{B \vee x \quad C \vee \overline{x}}{B \vee C}$$

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Resolution Derivation

Sequence of sets of clauses, or clause configurations, $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for clause $C \in F$ (axiom)

Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for clause $C \in \mathbb{C}_{t-1}$

Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{B \vee C\}$ for clause $B \vee C$ inferred by resolution rule from $B \vee x, C \vee \overline{x} \in \mathbb{C}_{t-1}$

Resolution derivation $\pi : F \vdash D$ of clause D from F:
Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $D \in \mathbb{C}_\tau$

Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Resolution Derivation

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Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $D \in \mathbb{C}_\tau$

Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Resolution Derivation

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Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $D \in \mathbb{C}_\tau$

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Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Resolution Derivation

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Resolution derivation $\pi : F \vdash D$ of clause D from F:
Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $D \in \mathbb{C}_\tau$

Resolution refutation of F:
Derivation $\pi : F \vdash 0$ of empty clause 0 from F

Derivation Length, Width and Space

- Length $L(\pi)$ of derivation $\pi : F \vdash D$
  # distinct clauses in all of π
- Width $W(\pi)$ of derivation $\pi : F \vdash D$
  # literals in largest clause in π
- Space $Sp(\pi)$ of derivation $\pi : F \vdash D$
  # clauses in largest clause configuration $\mathbb{C}_t \in \pi$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	0
Width so far	0
Space so far	0

$$\left[\quad \right]$$

Empty start configuration

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	1
Width so far	1
Space so far	1

$$\begin{bmatrix} p_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 1: p_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	2
Width so far	1
Space so far	2

$$\begin{bmatrix} p_1 \\ q_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 2: q_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	3
Width so far	3
Space so far	3

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Download axiom 5: $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	3
Width so far	3
Space so far	3

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \lor \overline{q}_1 \lor u_1 \\ \\ \end{bmatrix}$$

Infer $\overline{q}_1 \lor u_1$ from p_1 and $\overline{p}_1 \lor \overline{q}_1 \lor u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	4
Width so far	3
Space so far	4

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{q}_1 \vee u_1 \\ \end{bmatrix}$$

Infer $\overline{q}_1 \vee u_1$ from p_1 and $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	4
Width so far	3
Space so far	4

$$\left[\begin{array}{l} p_1 \\ q_1 \\ \overline{p}_1 \lor \overline{q}_1 \lor u_1 \\ \overline{q}_1 \lor u_1 \end{array}\right]$$

Erase clause $\overline{p}_1 \lor \overline{q}_1 \lor u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	4
Width so far	3
Space so far	4

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	4
Width so far	3
Space so far	4

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Erase clause p_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	4
Width so far	3
Space so far	4

$$\begin{bmatrix} q_1 \\ \overline{q}_1 \lor u_1 \\ \\ \\ \end{bmatrix}$$

Erase clause p_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	4
Width so far	3
Space so far	4

$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \\ \\ \\ \end{bmatrix}$$

Infer u_1 from q_1 and $\overline{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	5
Width so far	3
Space so far	4

$$\begin{bmatrix} q_1 \\ \overline{q}_1 \lor u_1 \\ u_1 \\ \\ \end{bmatrix}$$

Infer u_1 from q_1 and $\overline{q}_1 \lor u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	5
Width so far	3
Space so far	4

$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \\ u_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{q}_1 \vee u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	5
Width so far	3
Space so far	4

$$\begin{bmatrix} q_1 \\ u_1 \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{q}_1 \lor u_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	5
Width so far	3
Space so far	4

$$\begin{bmatrix} q_1 \\ u_1 \\ \\ \\ \end{bmatrix}$$

Erase clause q_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	5
Width so far	3
Space so far	4

$$\begin{bmatrix} u_1 \\ \\ \\ \end{bmatrix}$$

Erase clause q_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	6
Width so far	3
Space so far	4

$$\begin{bmatrix} u_1 \\ r_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 3: r_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	7
Width so far	3
Space so far	4

$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \\ \end{bmatrix}$$

Download axiom 4: s_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	8
Width so far	3
Space so far	4

$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \lor \overline{s}_1 \lor v_1 \\ \end{bmatrix}$$

Download axiom 6: $\overline{r}_1 \lor \overline{s}_1 \lor v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	8
Width so far	3
Space so far	4

$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \end{bmatrix}$$

Infer $\overline{s}_1 \vee v_1$ from r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	9
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{s}_1 \vee v_1 \end{bmatrix}$$

Infer $\overline{s}_1 \vee v_1$ from r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	9
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{s}_1 \vee v_1 \end{bmatrix}$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	9
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{s}_1 \lor v_1 \\ \end{bmatrix}$$

Erase clause $\overline{r}_1 \lor \overline{s}_1 \lor v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	9
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ r_1 \\ s_1 \\ \overline{s}_1 \lor v_1 \end{array}\right]$$

Erase clause r_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	9
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ s_1 \\ \overline{s}_1 \vee v_1 \\ \\ \end{array}\right]$$

Erase clause r_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	9
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \lor v_1 \\ \\ \end{bmatrix}$$

Infer v_1 from s_1 and $\overline{s}_1 \lor v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	10
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \vee v_1 \\ v_1 \end{bmatrix}$$

Infer v_1 from s_1 and $\overline{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	10
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \lor v_1 \\ v_1 \\ \end{bmatrix}$$

Erase clause $\overline{s}_1 \lor v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	10
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ s_1 \\ v_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{s}_1 \vee v_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	10
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ s_1 \\ v_1 \\ \\ \end{bmatrix}$$

Erase clause s_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	10
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ v_1 \\ \\ \\ \end{bmatrix}$$

Erase clause s_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	11
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Download axiom 7: $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	11
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \lor \overline{v}_1 \lor z_1 \\ \\ \end{bmatrix}$$

Infer $\overline{v}_1 \lor z_1$ from u_1 and $\overline{u}_1 \lor \overline{v}_1 \lor z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	12
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \lor \overline{v}_1 \lor z_1 \\ \overline{v}_1 \lor z_1 \end{bmatrix}$$

Infer $\overline{v}_1 \lor z_1$ from u_1 and $\overline{u}_1 \lor \overline{v}_1 \lor z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	12
Width so far	3
Space so far	5

$$\left[\begin{array}{l} u_1 \\ v_1 \\ \overline{u}_1 \lor \overline{v}_1 \lor z_1 \\ \overline{v}_1 \lor z_1 \end{array}\right]$$

Erase clause $\overline{u}_1 \lor \overline{v}_1 \lor z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	12
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	12
Width so far	3
Space so far	5

$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{v}_1 \lor z_1 \\ \\ \end{bmatrix}$$

Erase clause u_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	12
Width so far	3
Space so far	5

$$\left[\begin{array}{l} v_1 \\ \overline{v}_1 \vee z_1 \\ \\ \\ \end{array}\right]$$

Erase clause u_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	12
Width so far	3
Space so far	5

$$\begin{bmatrix} v_1 \\ \overline{v}_1 \lor z_1 \\ \\ \\ \end{bmatrix}$$

Infer z_1 from v_1 and $\overline{v}_1 \lor z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	13
Width so far	3
Space so far	5

$$\left[\begin{array}{l} v_1 \\ \overline{v}_1 \vee z_1 \\ z_1 \\ \\ \end{array}\right]$$

Infer z_1 from
v_1 and $\overline{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	13
Width so far	3
Space so far	5

$$\begin{bmatrix} v_1 \\ \overline{v}_1 \vee z_1 \\ z_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{v}_1 \vee z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	13
Width so far	3
Space so far	5

$$\begin{bmatrix} v_1 \\ z_1 \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{v}_1 \lor z_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \lor \overline{q}_1 \lor u_1$
6. $\overline{r}_1 \lor \overline{s}_1 \lor v_1$
7. $\overline{u}_1 \lor \overline{v}_1 \lor z_1$
8. $\overline{z}_1$

Length so far	13
Width so far	3
Space so far	5

$$\begin{bmatrix} v_1 \\ z_1 \\ \\ \\ \end{bmatrix}$$

Erase clause v_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	13
Width so far	3
Space so far	5

$$\begin{bmatrix} z_1 \\ \\ \\ \end{bmatrix}$$

Erase clause v_1

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	14
Width so far	3
Space so far	5

$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 8: $\overline{z}_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	14
Width so far	3
Space so far	5

$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \\ \end{bmatrix}$$

Infer 0 from z_1 and $\overline{z}_1$

Example Resolution Proof

1. p_1
2. q_1
3. r_1
4. s_1
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$
8. $\overline{z}_1$

Length so far	15
Width so far	3
Space so far	5

$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ 0 \end{bmatrix}$$

Infer 0 from
z_1 and $\overline{z}_1$

Length, Width and Space of Refuting F

- Length of refuting F is

$$L(F \vdash 0) = \min_{\pi : F \vdash 0} \{ L(\pi) \}$$

- Width of refuting F is

$$W(F \vdash 0) = \min_{\pi : F \vdash 0} \{ W(\pi) \}$$

- Space of refuting F is

$$Sp(F \vdash 0) = \min_{\pi : F \vdash 0} \{ Sp(\pi) \}$$

$$\begin{aligned} L(F \vdash 0) &\leq 2^{(\text{\# variables in } F + 1)} \\ W(F \vdash 0) &\leq \text{\# variables in } F \\ Sp(F \vdash 0) &\leq \min(\text{\# variables in } F, \text{\# clauses in } F) + \mathcal{O}(1) \end{aligned}$$

Length, Width and Space of Refuting F

- Length of refuting F is

$$L(F \vdash 0) = \min_{\pi : F \vdash 0} \{ L(\pi) \}$$

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$$W(F \vdash 0) = \min_{\pi : F \vdash 0} \{ W(\pi) \}$$

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$$Sp(F \vdash 0) = \min_{\pi : F \vdash 0} \{ Sp(\pi) \}$$

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Connection between Length and Width

A narrow resolution proof is necessarily short.
For a proof in width w, $\left(2 \cdot |Vars(F)|\right)^{w}$ is an upper bound on the number of possible clauses.

There is a kind of converse to this:

Theorem (Ben-Sasson & Wigderson 1999)

The width of refuting a k-CNF formula F over n variables is

$$W(F \vdash 0) = \mathcal{O}\left(\sqrt{n \log L(F \vdash 0)}\right).$$

This bound on width in terms of length is essentially optimal (Bonet & Galesi 1999).

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Connection between Space and Width

All previously shown lower bounds on space coincide with lower bounds on width—true in general?

Theorem (Atserias & Dalmau 2003)

For any unsatisfiable k-CNF formula F it holds that

$$Sp(F \vdash 0) \geq W(F \vdash 0) - \mathcal{O}(1).$$

But do space and width always coincide?
Or is there a k-CNF formula family $\{F_n\}_{n=1}^{\infty}$ such that $Sp(F_n \vdash 0) = \omega(W(F_n \vdash 0))$?

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Pebbles Games on Graphs

One-player game played on directed acyclic graphs (DAGs)

- Devised for studying programming languages and compiler construction
- Have found a variety of applications in complexity theory

Conventions

- $V(G)$ denote the vertices of a DAG G
- vertices with indegree 0 are sources
- vertices with outdegree 0 are targets

Only consider DAGs with single target z and and indegree 2 for all non-source vertices

Formal Definition of Pebble Game

Pebble configuration: pair of subsets $\mathbb{P} = (B, W)$ of black- and white-pebbled vertices

Black-white pebbling: sequence $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ such that $\mathbb{P}_0 = (\emptyset, \emptyset)$ and $\mathbb{P}_t$ follows from $\mathbb{P}_{t-1}$ by one of the rules:

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all all immediate predecessors have pebbles on them

Goal: reach $\mathbb{P}_\tau = (\{z\}, \emptyset)$ using few pebbles

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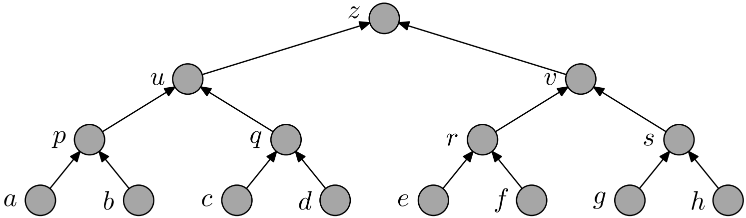
Black-White Pebbling of Binary Tree of Height 3



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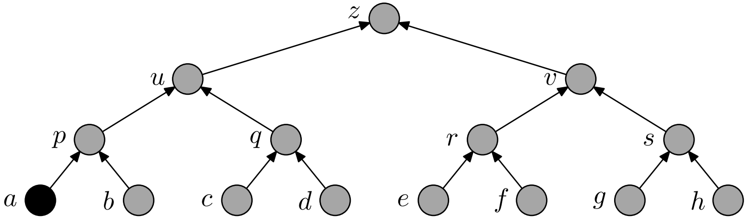

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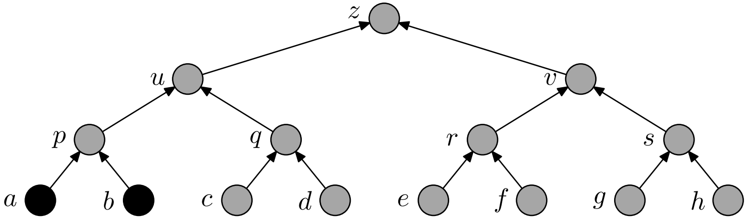

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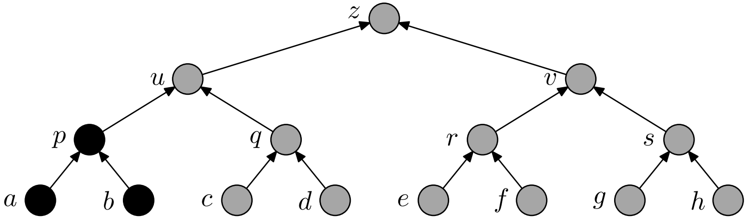

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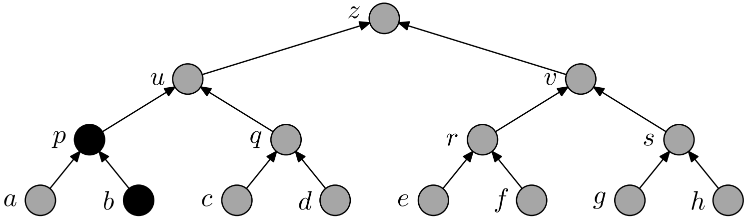

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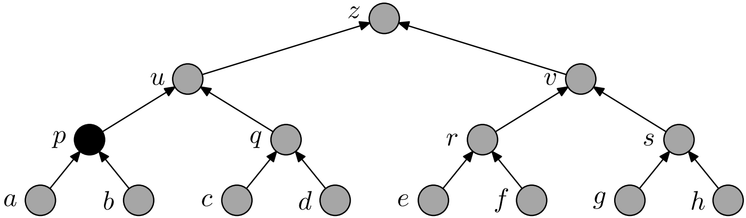

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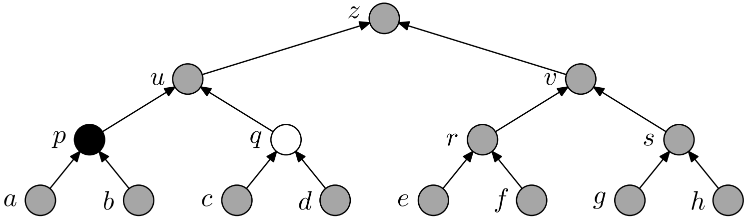

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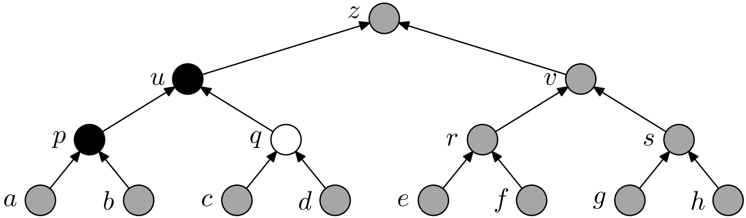

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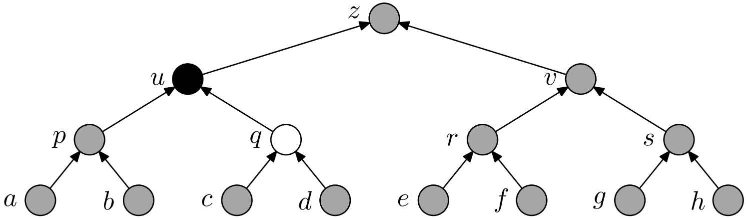

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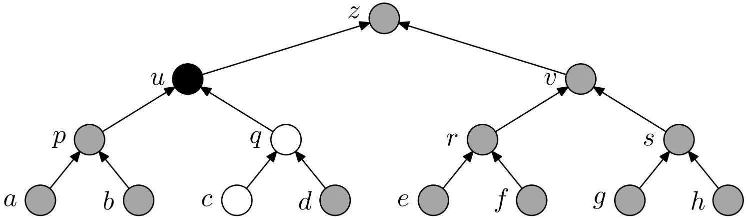

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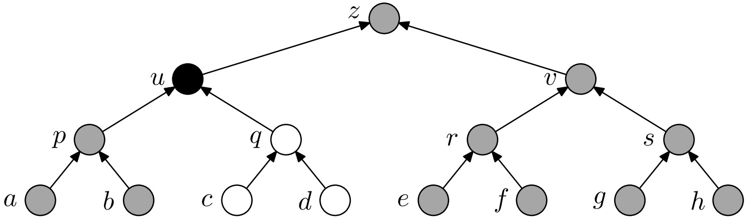

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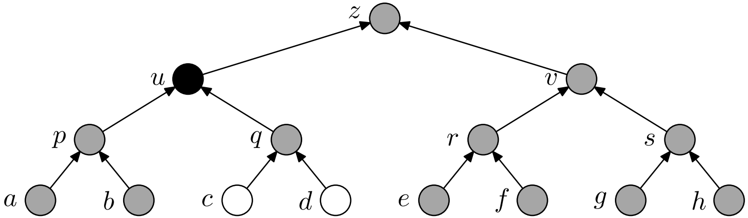

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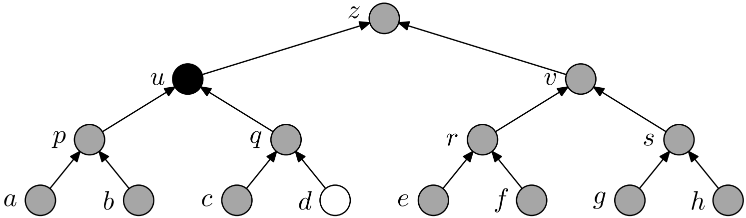

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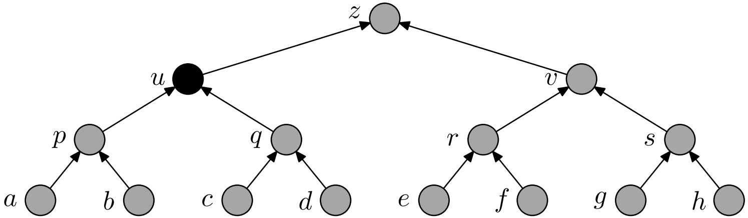

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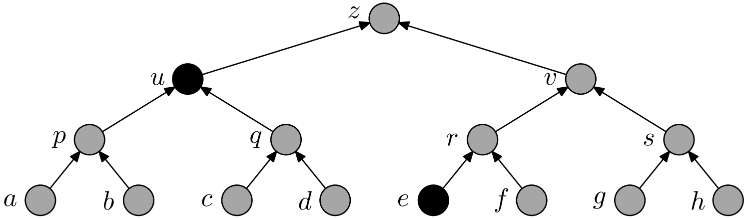

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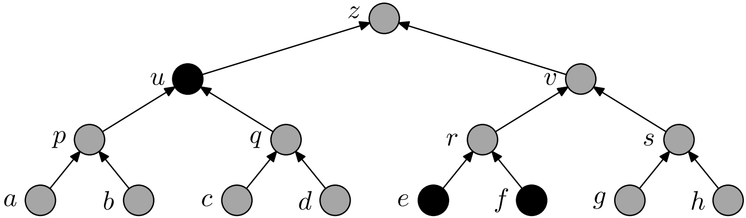

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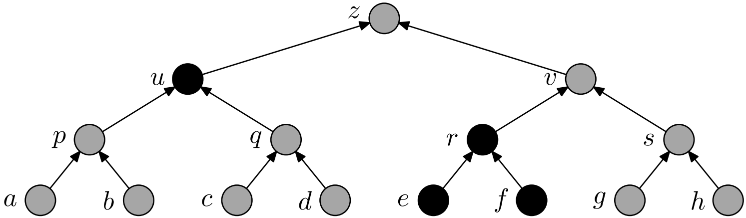

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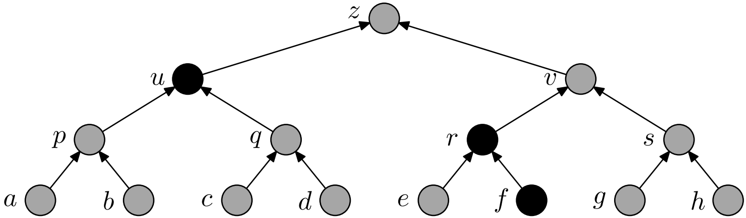

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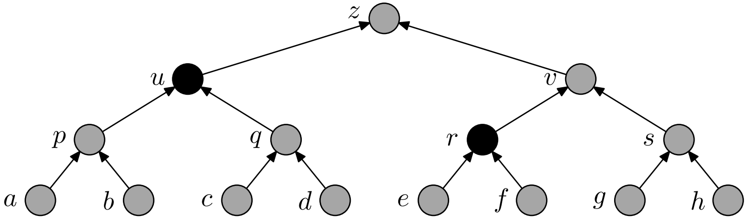

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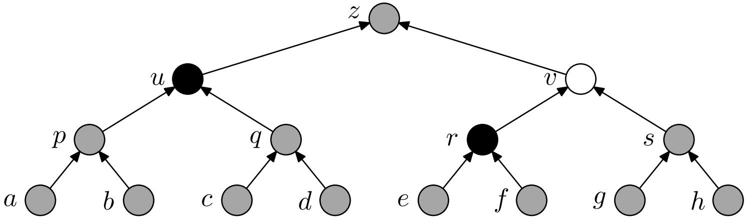
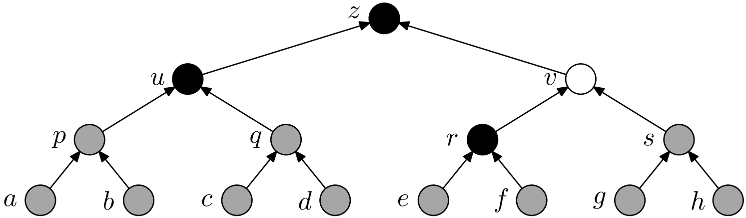


Black-White Pebbling of Binary Tree of Height 3



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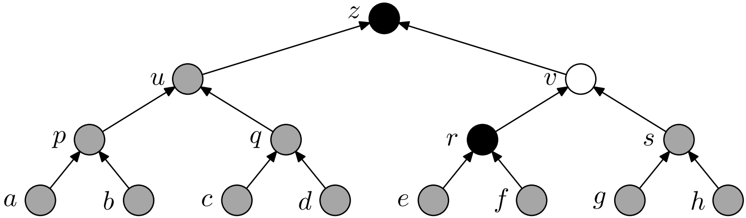

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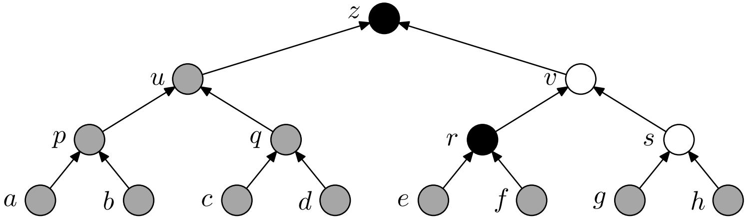

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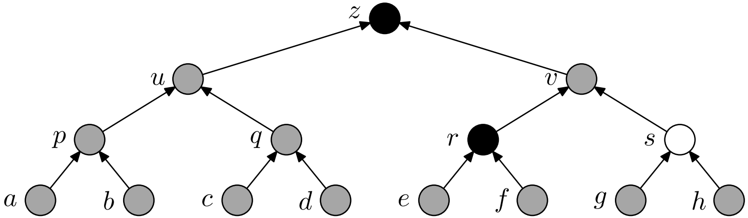

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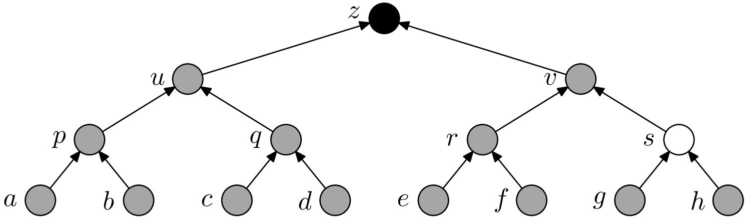

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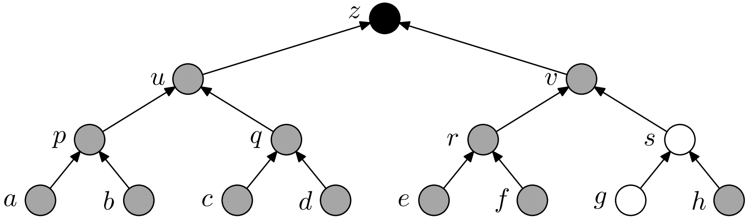

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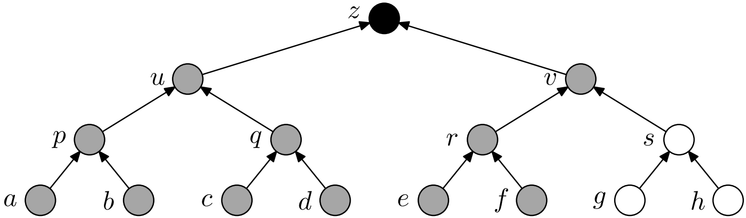

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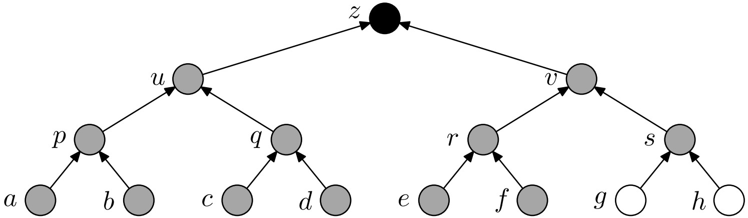

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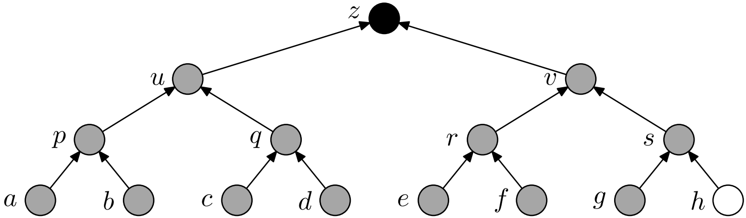

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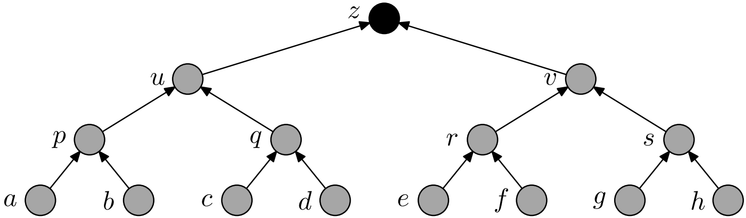
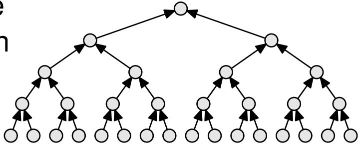


Pebbling Price

- Cost of pebbling $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$:
  max # pebbles in any $\mathbb{P}_t = (B_t, W_t)$
- Black-white pebbling price *BW-Peb*(G) of DAG G is minimal cost of any pebbling reaching $(\{z\}, \emptyset)$
- (Black) pebbling price *Peb*(G) is minimal cost of any pebbling reaching $(\{z\}, \emptyset)$ using black pebbles only ($W_t = \emptyset$ for all t)

Pebbling Price of Binary Trees

Let T_h denote complete binary tree of height h considered as DAG with edges directed towards root



- Pebbling price of T_h is

$$Peb(T_h) = h + 2$$

  (easy induction over the tree height)

- Black-white pebbling price is

$$BW\text{-}Peb(T_h) = \left\lfloor \frac{h}{2} \right\rfloor + 3 = \Omega(h)$$

  (Lengauer & Tarjan 1980)

Definition of Pebbling Contradiction

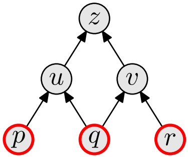
CNF formula encoding pebble game on DAG G with sources S, unique target z and all non-source vertices having indegree 2

Associate d variables $v_1, \ldots, v_d$ with every vertex $v \in V(G)$

The dth degree pebbling contradiction Peb^d_G over G is the conjunction of the following (written as clauses):

- $\bigvee_{i=1}^{d} s_i$ for all $s \in S$ (source axioms)
- $\left(\bigvee_{i=1}^{d} u_i \wedge \bigvee_{j=1}^{d} v_j\right) \rightarrow \bigvee_{l=1}^{d} w_l$ for all $w \in V(G) \setminus S$, where u, v are the two predecessors of w (pebbling axioms)
- $\bigwedge_{i=1}^{d} \overline{z}_i$ (target axioms)

Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



$$
\begin{aligned}
& (p_1 \lor p_2) \\
\land\ & (q_1 \lor q_2) \\
\land\ & (r_1 \lor r_2) \\
\land\ & (\overline{p}_1 \lor \overline{q}_1 \lor u_1 \lor u_2) \\
\land\ & (\overline{p}_1 \lor \overline{q}_2 \lor u_1 \lor u_2) \\
\land\ & (\overline{p}_2 \lor \overline{q}_1 \lor u_1 \lor u_2) \\
\land\ & (\overline{p}_2 \lor \overline{q}_2 \lor u_1 \lor u_2) \\
\land\ & (\overline{q}_1 \lor \overline{r}_1 \lor v_1 \lor v_2) \\
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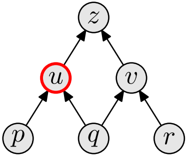
\land\ & (\overline{u}_1 \lor \overline{v}_2 \lor z_1 \lor z_2) \\
\land\ & (\overline{u}_2 \lor \overline{v}_1 \lor z_1 \lor z_2) \\
\land\ & (\overline{u}_2 \lor \overline{v}_2 \lor z_1 \lor z_2) \\
\land\ & \overline{z}_1 \\
\land\ & \overline{z}_2
\end{aligned}
$$

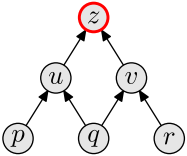
Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



$$
\begin{aligned}
&(p_1 \lor p_2)\\
\land\ &(q_1 \lor q_2)\\
\land\ &(r_1 \lor r_2)\\
\land\ &(\overline{p}_1 \lor \overline{q}_1 \lor u_1 \lor u_2)\\
\land\ &(\overline{p}_1 \lor \overline{q}_2 \lor u_1 \lor u_2)\\
\land\ &(\overline{p}_2 \lor \overline{q}_1 \lor u_1 \lor u_2)\\
\land\ &(\overline{p}_2 \lor \overline{q}_2 \lor u_1 \lor u_2)\\
\land\ &(\overline{q}_1 \lor \overline{r}_1 \lor v_1 \lor v_2)\\
\land\ &(\overline{q}_1 \lor \overline{r}_2 \lor v_1 \lor v_2)
\end{aligned}
$$

$$
\begin{aligned}
\land\ &(\overline{q}_2 \lor \overline{r}_1 \lor v_1 \lor v_2)\\
\land\ &(\overline{q}_2 \lor \overline{r}_2 \lor v_1 \lor v_2)\\
\land\ &(\overline{u}_1 \lor \overline{v}_1 \lor z_1 \lor z_2)\\
\land\ &(\overline{u}_1 \lor \overline{v}_2 \lor z_1 \lor z_2)\\
\land\ &(\overline{u}_2 \lor \overline{v}_1 \lor z_1 \lor z_2)\\
\land\ &(\overline{u}_2 \lor \overline{v}_2 \lor z_1 \lor z_2)\\
\land\ &\overline{z}_1\\
\land\ &\overline{z}_2
\end{aligned}
$$

Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



$$
\begin{aligned}
& (p_1 \lor p_2) \\
\land\ & (q_1 \lor q_2) \\
\land\ & (r_1 \lor r_2) \\
\land\ & (\overline{p}_1 \lor \overline{q}_1 \lor u_1 \lor u_2) \\
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\land\ & \overline{z}_1 \\
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\end{aligned}
$$

Pebbling Contradictions Easy w.r.t. Length and Width

Peb^d_G is an unsatisfiable (2+d)-CNF formula with

- $d \cdot |V(G)|$ variables
- $\mathcal{O}\big(d^2 \cdot |V(G)|\big)$ clauses

Can be refuted by deriving $\bigvee_{i=1}^{d} v_i$ for all $v \in V(G)$ inductively in topological order and resolving with target axioms $\overline{z}_i$, $i \in [d]$

It follows that

- $L(F \vdash 0) \;=\; \mathcal{O}\big(d^2 \cdot |V(G)|\big)$
- $W(F \vdash 0) \;=\; \mathcal{O}(d)$

(Ben-Sasson et al. 2000)

What about Pebbling Contradictions and Space?

Upper bounds:

- Arbitrary DAGs G
  optimal black pebbling of G + proof from previous slide:
  $Sp\left(Peb_G^d \vdash 0\right) \leq Peb(G) + \mathcal{O}(1)$
- Binary trees T_h
  improvement by Esteban & Torán (2003):
  $Sp\left(Peb_{T_h}^2 \vdash 0\right) \leq \left\lceil \frac{2h+1}{3} \right\rceil + 3 = \frac{2}{3} Peb(T_h) + \mathcal{O}(1)$
- Only one variable / vertex
  Ben-Sasson (2002):
  $Sp\left(Peb_G^1 \vdash 0\right) = \mathcal{O}(1)$ for arbitrary G

No lower bounds on space for $d \geq 2$ previously known

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Our Results

Theorem

Let $Peb^d_{T_h}$ denote the pebbling contradiction of degree $d \geq 2$ defined over the complete binary tree of height h. Then the space of refuting $Peb^d_{T_h}$ in resolution is $Sp\left(Peb^d_{T_h} \vdash 0\right) = \Theta(h)$.

Corollary

For all $k \geq 4$, there is a family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and refutation space $Sp(F_n \vdash 0) = \Theta(\log n)$.

Proof Idea

Prove lower bounds on space of $\pi : Peb^d_G \vdash 0$ by

1. Interpreting set of clauses $\mathbb{C}_t \in \pi$ in terms of black and white pebbles on G
2. Showing that if $\mathbb{C}_t$ induces N black and white pebbles it contains at least N clauses (if $d \geq 2$)
3. Establishing that $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ induces black-white pebbling $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ (works only for binary trees T_h)

Then some $\mathbb{C}_t \in \pi$ must induce $\textit{BW-Peb}(T_h)$ pebbles

$$\Downarrow$$

$$|\mathbb{C}_t| \geq \textit{BW-Peb}(T_h) = \Omega(h)$$

$$\Downarrow$$

$$Sp\left(Peb^d_{T_h} \vdash 0\right) = \Omega(h)$$

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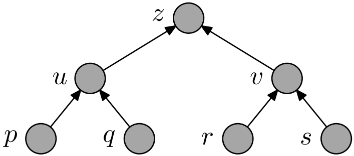
$$\Downarrow$$

$$Sp\left(Peb^d_{T_h} \vdash 0\right) = \Omega(h)$$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target



$$\left[\quad \right]$$

Empty start configuration

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
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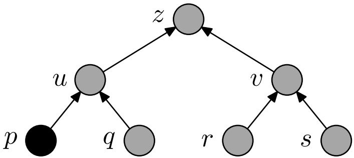

$$\begin{bmatrix} p_1 \\ \\ \\ \\ \end{bmatrix}$$

Download axiom 1: p_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
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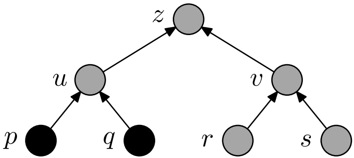
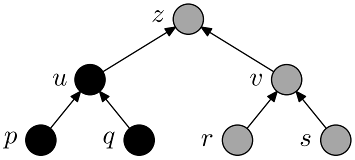


$$\begin{bmatrix} p_1 \\ q_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 2: q_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target



$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Download axiom 5: $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
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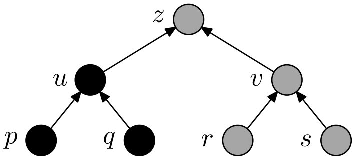

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Infer $\overline{q}_1 \vee u_1$ from p_1 and $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
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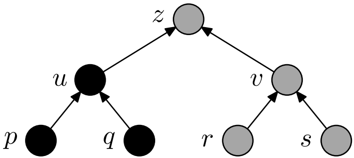

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{q}_1 \vee u_1 \end{bmatrix}$$

Infer $\overline{q}_1 \vee u_1$ from p_1 and $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
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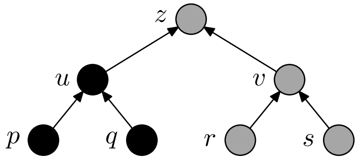

$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

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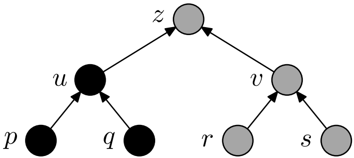
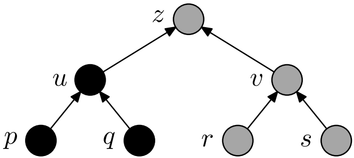


$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{q}_1 \lor u_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{p}_1 \lor \overline{q}_1 \lor u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

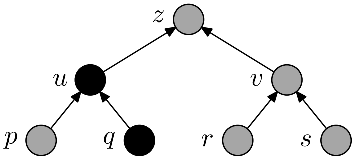


$$\begin{bmatrix} p_1 \\ q_1 \\ \overline{q}_1 \vee u_1 \\ \\ \end{bmatrix}$$

Erase clause p_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

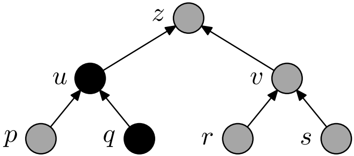


$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \\ \\ \\ \end{bmatrix}$$

Erase clause p_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

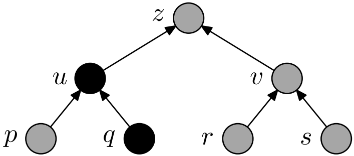


$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \\ \\ \\ \end{bmatrix}$$

Infer u_1 from q_1 and $\overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

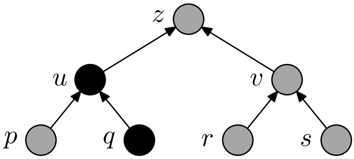


$$\left[\begin{array}{l} q_1 \\ \overline{q}_1 \vee u_1 \\ u_1 \end{array}\right]$$

Infer u_1 from q_1 and $\overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

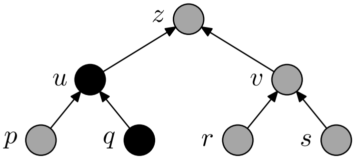


$$\begin{bmatrix} q_1 \\ \overline{q}_1 \vee u_1 \\ u_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{q}_1 \vee u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

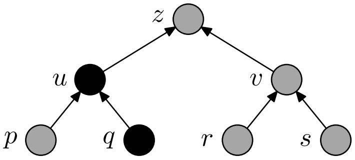


$$\begin{bmatrix} q_1 \\ u_1 \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{q}_1 \lor u_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

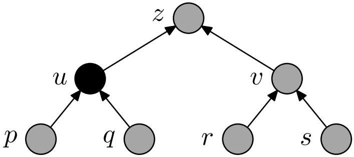


$$\begin{bmatrix} q_1 \\ u_1 \\ \\ \\ \end{bmatrix}$$

Erase clause q_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

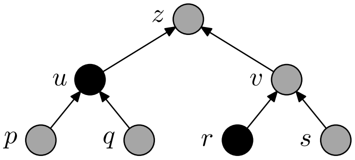


$$\begin{bmatrix} u_1 \\ \\ \\ \\ \end{bmatrix}$$

Erase clause q_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

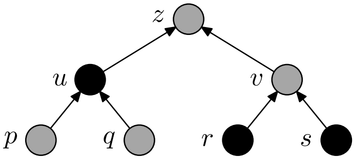


$$\begin{bmatrix} u_1 \\ r_1 \end{bmatrix}$$

Download axiom 3: r_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

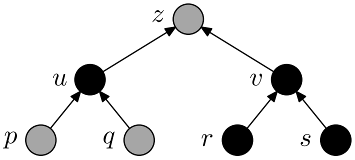


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \end{bmatrix}$$

Download axiom 4: s_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

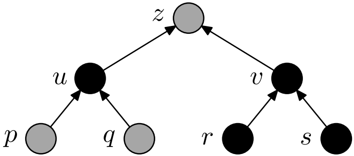


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \end{bmatrix}$$

Download axiom 6: $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

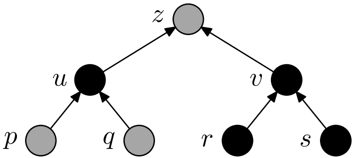


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \end{bmatrix}$$

Infer $\overline{s}_1 \vee v_1$ from r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

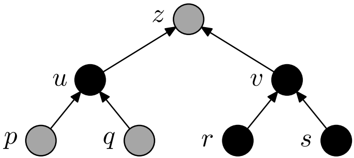


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{s}_1 \vee v_1 \end{bmatrix}$$

Infer $\overline{s}_1 \vee v_1$ from r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

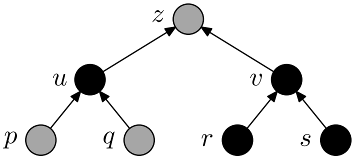


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \overline{s}_1 \vee v_1 \end{bmatrix}$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

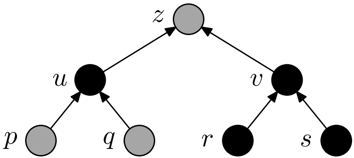


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{s}_1 \lor v_1 \end{bmatrix}$$

Erase clause $\overline{r}_1 \lor \overline{s}_1 \lor v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

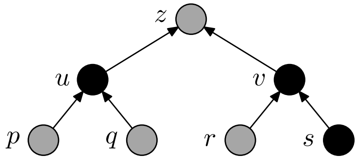


$$\begin{bmatrix} u_1 \\ r_1 \\ s_1 \\ \overline{s}_1 \lor v_1 \end{bmatrix}$$

Erase clause r_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

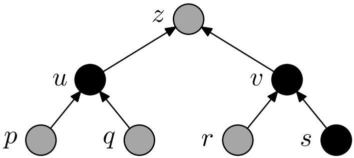


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \lor v_1 \\ \\ \end{bmatrix}$$

Erase clause r_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

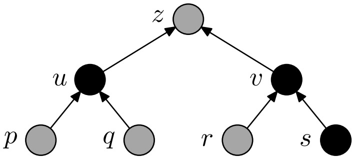


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \lor v_1 \\ \\ \end{bmatrix}$$

Infer v_1 from s_1 and $\overline{s}_1 \lor v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

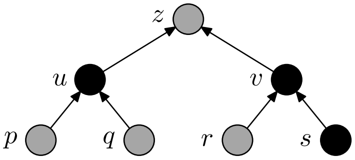


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \lor v_1 \\ \color{red}{v_1} \end{bmatrix}$$

Infer v_1 from
s_1 and $\overline{s}_1 \lor v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

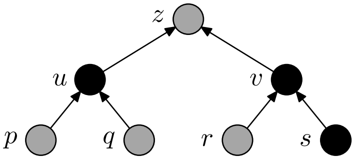


$$\begin{bmatrix} u_1 \\ s_1 \\ \overline{s}_1 \vee v_1 \\ v_1 \end{bmatrix}$$

Erase clause $\overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

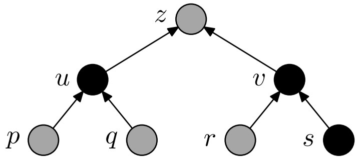


$$\begin{bmatrix} u_1 \\ s_1 \\ v_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{s}_1 \vee v_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

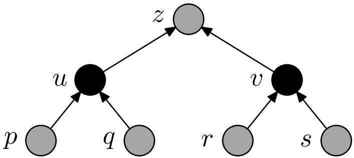


$$\begin{bmatrix} u_1 \\ s_1 \\ v_1 \end{bmatrix}$$

Erase clause s_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

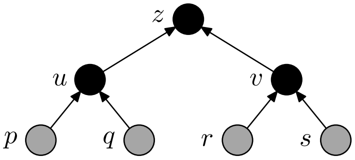


$$\begin{bmatrix} u_1 \\ v_1 \\ \\ \\ \end{bmatrix}$$

Erase clause s_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

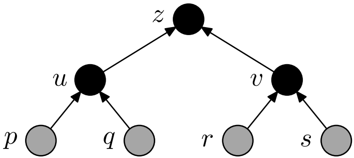


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Download axiom 7: $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

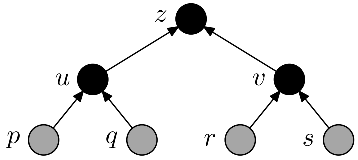


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \lor \overline{v}_1 \lor z_1 \\ \\ \end{bmatrix}$$

Infer $\overline{v}_1 \lor z_1$ from u_1 and $\overline{u}_1 \lor \overline{v}_1 \lor z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

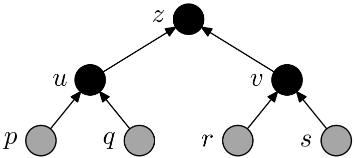


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \lor \overline{v}_1 \lor z_1 \\ \overline{v}_1 \lor z_1 \end{bmatrix}$$

Infer $\overline{v}_1 \lor z_1$ from u_1 and $\overline{u}_1 \lor \overline{v}_1 \lor z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

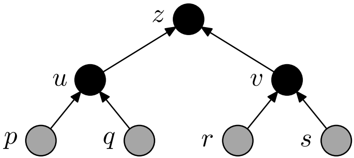


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{v}_1 \vee z_1 \\ \end{bmatrix}$$

Erase clause $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

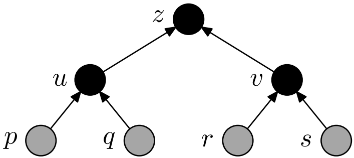


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{v}_1 \vee z_1 \\ \\ \end{bmatrix}$$

Erase clause $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

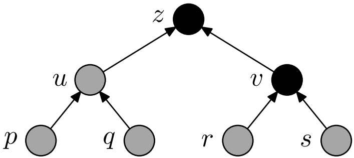


$$\begin{bmatrix} u_1 \\ v_1 \\ \overline{v}_1 \lor z_1 \\ \\ \end{bmatrix}$$

Erase clause u_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

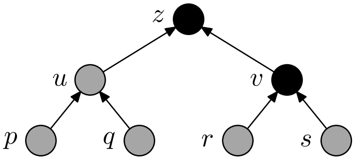


$$\begin{bmatrix} v_1 \\ \overline{v}_1 \vee z_1 \\ \\ \\ \end{bmatrix}$$

Erase clause u_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

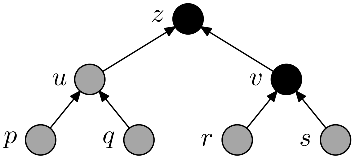


$$\begin{bmatrix} v_1 \\ \overline{v}_1 \vee z_1 \end{bmatrix}$$

Infer z_1 from v_1 and $\overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

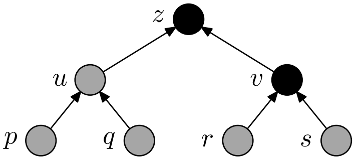


$$\begin{bmatrix} v_1 \\ \overline{v}_1 \lor z_1 \\ z_1 \end{bmatrix}$$

Infer z_1 from v_1 and $\overline{v}_1 \lor z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

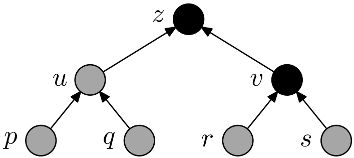


$$\begin{bmatrix} v_1 \\ \overline{v}_1 \vee z_1 \\ z_1 \end{bmatrix}$$

Erase clause $\overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

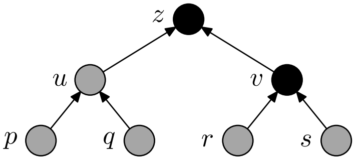


$$\begin{bmatrix} v_1 \\ z_1 \\ \\ \\ \end{bmatrix}$$

Erase clause $\overline{v}_1 \vee z_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

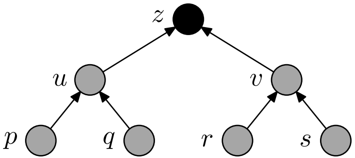


$$\begin{bmatrix} v_1 \\ z_1 \\ \\ \\ \end{bmatrix}$$

Erase clause v_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

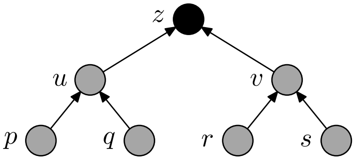


$$\begin{bmatrix} z_1 \\ \\ \\ \\ \end{bmatrix}$$

Erase clause v_1

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

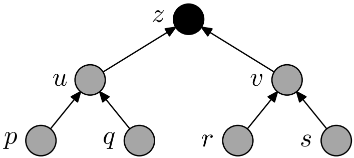


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \\ \end{bmatrix}$$

Download axiom 8: $\overline{z}_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

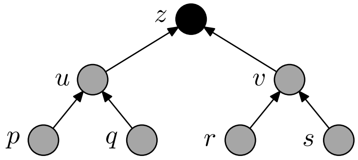


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \\ \\ \end{bmatrix}$$

Infer 0 from z_1 and $\overline{z}_1$

Developing an Intuition for Black Pebbles

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target



$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ 0 \end{bmatrix}$$

Infer 0 from z_1 and $\overline{z}_1$

Intuition for Black and White Pebbles

Induced Black Pebble

$\mathbb{C}_t \vDash \bigvee_{i=1}^{d} v_i \Leftrightarrow$ black pebble on v with no white pebbles below

How to interpret white pebbles on W below black pebble v?
Getting white pebbles *off* vertices is exactly as hard as getting black pebbles *on* vertices

Assuming we could remove white pebbles from $W \Leftrightarrow$ place black pebbles on W, would have single black pebble on v left

Induced White Pebbles

$\mathbb{C}_t$ should induce white pebbles on W below v if assuming black pebbles on W, we get single black pebble on v
That is, if $\mathbb{C}_t \cup \left\{ \bigvee_{i=1}^{d} w_i \mid w \in W \right\} \vDash \bigvee_{i=1}^{d} v_i$.

Intuition for Black and White Pebbles

Induced Black Pebble

$\mathbb{C}_t \vDash \bigvee_{i=1}^{d} v_i \Leftrightarrow$ black pebble on v with no white pebbles below

How to interpret white pebbles on W below black pebble v? Getting white pebbles *off* vertices is exactly as hard as getting black pebbles *on* vertices

Assuming we could remove white pebbles from $W \Leftrightarrow$ place black pebbles on W, would have single black pebble on v left

Induced White Pebbles

$\mathbb{C}_t$ should induce white pebbles on W below v if assuming black pebbles on W, we get single black pebble on v
That is, if $\mathbb{C}_t \cup \left\{ \bigvee_{i=1}^{d} w_i \mid w \in W \right\} \vDash \bigvee_{i=1}^{d} v_i$.

Intuition for Black and White Pebbles

Induced Black Pebble

$\mathbb{C}_t \vDash \bigvee_{i=1}^{d} v_i \Leftrightarrow$ black pebble on v with no white pebbles below

How to interpret white pebbles on W below black pebble v?
Getting white pebbles *off* vertices is exactly as hard as getting black pebbles *on* vertices

Assuming we could remove white pebbles from $W \Leftrightarrow$ place black pebbles on W, would have single black pebble on v left

Induced White Pebbles

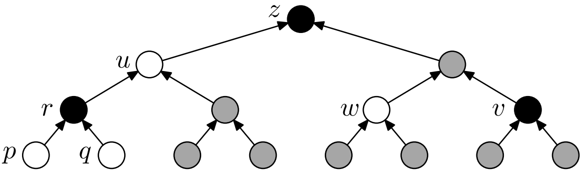
$\mathbb{C}_t$ should induce white pebbles on W below v if assuming black pebbles on W, we get single black pebble on v
That is, if $\mathbb{C}_t \cup \left\{ \bigvee_{i=1}^{d} w_i \mid w \in W \right\} \vDash \bigvee_{i=1}^{d} v_i$.

Example of Induced Pebble Subconfigurations

As an example, we would like the clause configuration

$$\mathbb{C} = \left[\begin{array}{ll} \overline{u}_i \vee \overline{w}_j \vee \bigvee_{l=1}^{d} z_l & \mid\; 1 \leq i,j \leq d \\ \overline{p}_i \vee \overline{q}_j \vee \bigvee_{l=1}^{d} r_l & \mid\; 1 \leq i,j \leq d \\ \bigvee_{l=1}^{d} v_l & \end{array} \right]$$

to induce the pebbles



Induced Pebbles and Clause Configuration Size

- Formalizing this yields interpretation of clause configuration $\mathbb{C}_t$ derived from Peb_G^d in terms of pebbles on G
- Hope that resolution proof $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ will correspond to black-white pebbling $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ of G under this interpretation
- But to get lower bound on space from this we need to show that

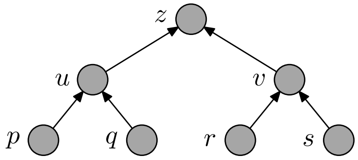
$\mathbb{C}_t$ induces many pebbles

$\Downarrow$

$\mathbb{C}_t$ contains many clauses

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

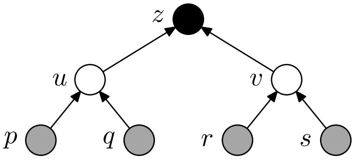


$$\left[\quad \right]$$

Empty start configuration

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

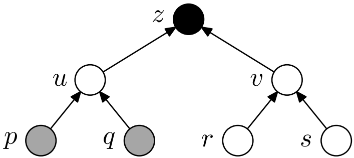


$$\left[\; \overline{u}_1 \lor \overline{v}_1 \lor z_1 \; \right]$$

Download axiom 7: $\overline{u}_1 \lor \overline{v}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

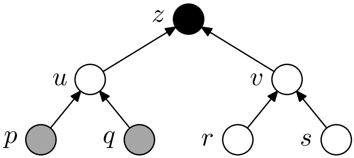


$$\begin{bmatrix} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \\ \end{bmatrix}$$

Download axiom 6: $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

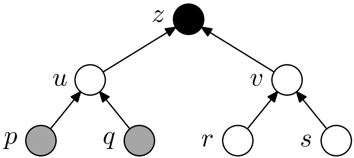


$$\left[\begin{array}{l} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee v_1 \end{array}\right]$$

Infer $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$ from $\overline{r}_1 \vee \overline{s}_1 \vee v_1$ and $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

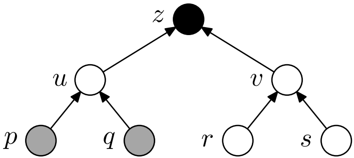


$$\begin{bmatrix} \overline{u}_1 \lor \overline{v}_1 \lor z_1 \\ \overline{r}_1 \lor \overline{s}_1 \lor v_1 \\ \overline{r}_1 \lor \overline{s}_1 \lor \overline{u}_1 \lor z_1 \end{bmatrix}$$

Infer $\overline{r}_1 \lor \overline{s}_1 \lor \overline{u}_1 \lor z_1$ from $\overline{r}_1 \lor \overline{s}_1 \lor v_1$ and $\overline{u}_1 \lor \overline{v}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

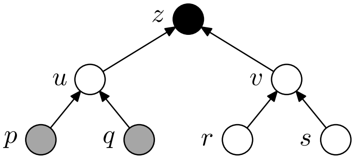


$$\begin{bmatrix} \overline{u}_1 \lor \overline{v}_1 \lor z_1 \\ \overline{r}_1 \lor \overline{s}_1 \lor v_1 \\ \overline{r}_1 \lor \overline{s}_1 \lor \overline{u}_1 \lor z_1 \end{bmatrix}$$

Erase clause $\overline{r}_1 \lor \overline{s}_1 \lor v_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

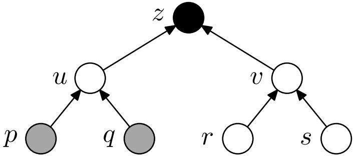


$$\left[\begin{array}{l} \overline{u}_1 \vee \overline{v}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \end{array} \right]$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee v_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

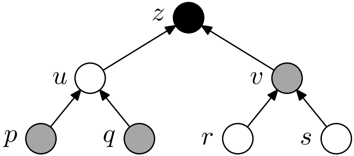


$$\begin{bmatrix} \overline{u}_1 \lor \overline{v}_1 \lor z_1 \\ \overline{r}_1 \lor \overline{s}_1 \lor \overline{u}_1 \lor z_1 \end{bmatrix}$$

Erase clause $\overline{u}_1 \lor \overline{v}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

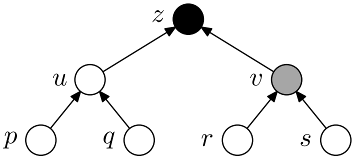


$$\left[\ \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1\ \right]$$

Erase clause $\overline{u}_1 \vee \overline{v}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

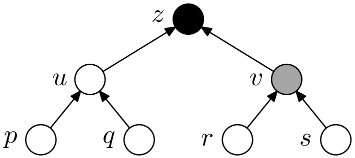


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \end{bmatrix}$$

Download axiom 5: $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

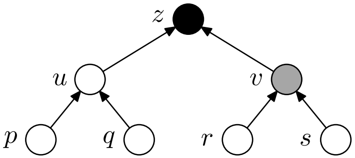


$$\begin{bmatrix} \overline{r}_1 \lor \overline{s}_1 \lor \overline{u}_1 \lor z_1 \\ \overline{p}_1 \lor \overline{q}_1 \lor u_1 \\ \ \end{bmatrix}$$

Infer $\overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1$ from $\overline{p}_1 \lor \overline{q}_1 \lor u_1$ and $\overline{r}_1 \lor \overline{s}_1 \lor \overline{u}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

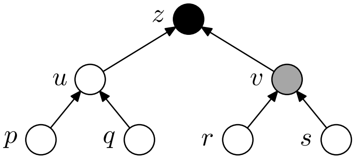


$$\begin{bmatrix} \overline{r}_1 \lor \overline{s}_1 \lor \overline{u}_1 \lor z_1 \\ \overline{p}_1 \lor \overline{q}_1 \lor u_1 \\ \overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \end{bmatrix}$$

Infer $\overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1$ from $\overline{p}_1 \lor \overline{q}_1 \lor u_1$ and $\overline{r}_1 \lor \overline{s}_1 \lor \overline{u}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1. p_1 Source
2. q_1 Source
3. r_1 Source
4. s_1 Source
5. $\overline{p}_1 \vee \overline{q}_1 \vee u_1$ Pebbling
6. $\overline{r}_1 \vee \overline{s}_1 \vee v_1$ Pebbling
7. $\overline{u}_1 \vee \overline{v}_1 \vee z_1$ Pebbling
8. $\overline{z}_1$ Target

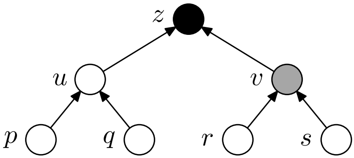


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee u_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

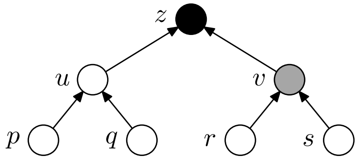


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \end{array}\right]$$

Erase clause $\overline{p}_1 \vee \overline{q}_1 \vee u_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

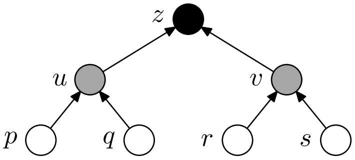


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1 \\ \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

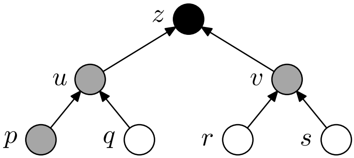


$$\left[\begin{array}{c} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \\ \end{array} \right]$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee \overline{u}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

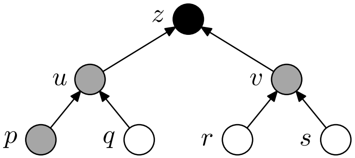


$$\begin{bmatrix} \overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ p_1 \\ \end{bmatrix}$$

Download axiom 1: p_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

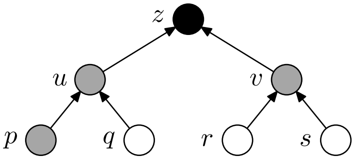


$$\begin{bmatrix} \overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ p_1 \\ \\ \end{bmatrix}$$

Infer $\overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1$ from p_1 and $\overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

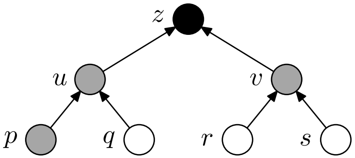


$$\left[\begin{array}{l} \overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ p_1 \\ \color{red}{\overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1} \end{array}\right]$$

Infer $\overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1$ from p_1 and $\overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

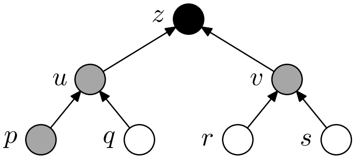


$$\begin{bmatrix} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ p_1 \\ \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause p_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

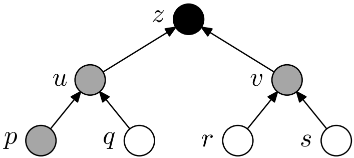


$$\begin{bmatrix} \overline{p}_1 \vee \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \end{bmatrix}$$

Erase clause p_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

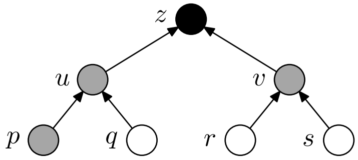


$$\left[\begin{array}{l} \overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ \end{array} \right]$$

Erase clause
$\overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

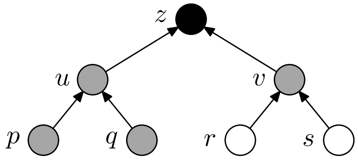


$$\left[\; \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \; \right]$$

Erase clause
$\overline{p}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

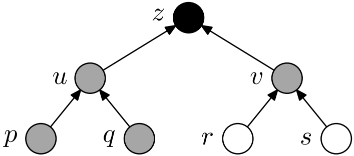


$$\begin{bmatrix} \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ q_1 \\ \end{bmatrix}$$

Download axiom 2: q_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

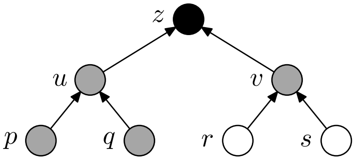


$$\begin{bmatrix} \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ q_1 \\ \end{bmatrix}$$

Infer $\overline{r}_1 \lor \overline{s}_1 \lor z_1$ from q_1 and $\overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

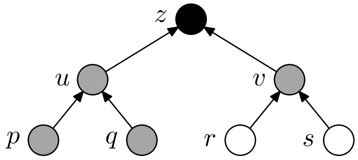


$$\begin{bmatrix} \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ q_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{r}_1 \vee \overline{s}_1 \vee z_1$ from q_1 and $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

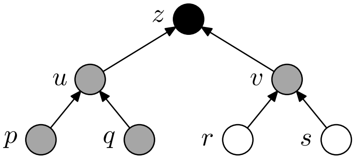


$$\begin{bmatrix} \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ q_1 \\ \overline{r}_1 \lor \overline{s}_1 \lor z_1 \end{bmatrix}$$

Erase clause q_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

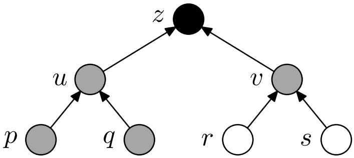


$$\left[\begin{array}{l} \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ \overline{r}_1 \lor \overline{s}_1 \lor z_1 \\ \end{array}\right]$$

Erase clause q_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

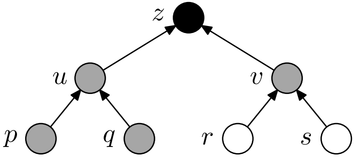


$$\left[\begin{array}{l} \overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \overline{r}_1 \vee \overline{s}_1 \vee z_1 \end{array}\right]$$

Erase clause $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

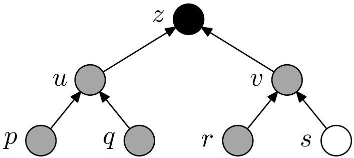


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \\ \end{array} \right]$$

Erase clause $\overline{q}_1 \vee \overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

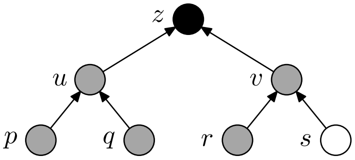


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \end{array}\right]$$

Download axiom 3: r_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

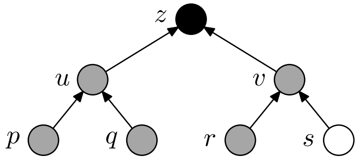


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \end{array}\right]$$

Infer $\overline{s}_1 \vee z_1$ from r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

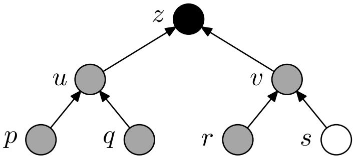


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \overline{s}_1 \vee z_1 \end{bmatrix}$$

Infer $\overline{s}_1 \vee z_1$ from r_1 and $\overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

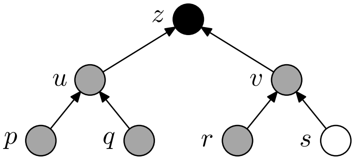


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ r_1 \\ \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause r_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

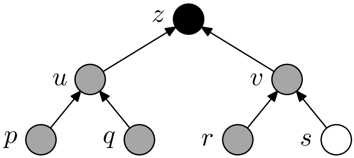


$$\left[\begin{array}{l} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \overline{s}_1 \vee z_1 \\ \end{array}\right]$$

Erase clause r_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

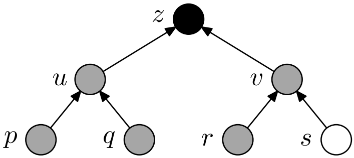


$$\begin{bmatrix} \overline{r}_1 \vee \overline{s}_1 \vee z_1 \\ \overline{s}_1 \vee z_1 \end{bmatrix}$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

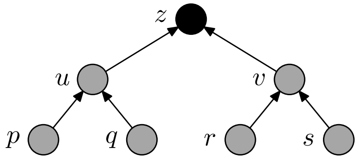


$$\left[\overline{s}_1 \vee z_1 \right]$$

Erase clause $\overline{r}_1 \vee \overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

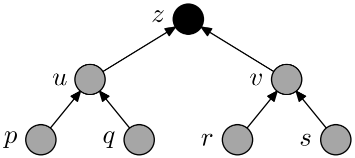


$$\begin{bmatrix} \overline{s}_1 \lor z_1 \\ s_1 \\ \end{bmatrix}$$

Download axiom 4: s_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

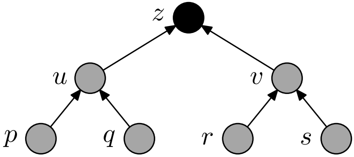


$$\begin{bmatrix} \overline{s}_1 \vee z_1 \\ s_1 \\ \end{bmatrix}$$

Infer z_1 from s_1 and $\overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

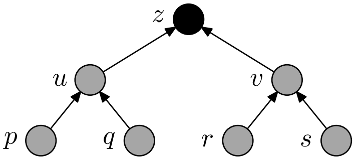


$$\begin{bmatrix} \overline{s}_1 \lor z_1 \\ s_1 \\ z_1 \end{bmatrix}$$

Infer z_1 from
s_1 and $\overline{s}_1 \lor z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \lor \overline{q}_1 \lor u_1$	Pebbling
6.	$\overline{r}_1 \lor \overline{s}_1 \lor v_1$	Pebbling
7.	$\overline{u}_1 \lor \overline{v}_1 \lor z_1$	Pebbling
8.	$\overline{z}_1$	Target

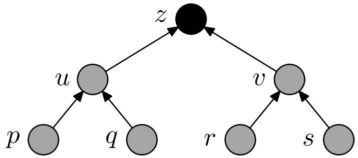


$$\begin{bmatrix} \overline{s}_1 \lor z_1 \\ s_1 \\ z_1 \end{bmatrix}$$

Erase clause s_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

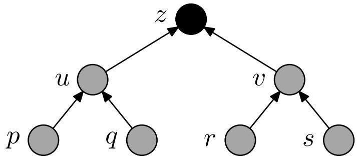


$$\begin{bmatrix} \overline{s}_1 \vee z_1 \\ z_1 \\ \end{bmatrix}$$

Erase clause s_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

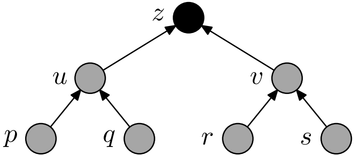


$$\begin{bmatrix} \overline{s}_1 \vee z_1 \\ z_1 \end{bmatrix}$$

Erase clause $\overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

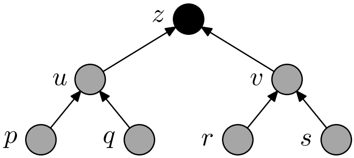


$$\begin{bmatrix} z_1 \end{bmatrix}$$

Erase clause $\overline{s}_1 \vee z_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

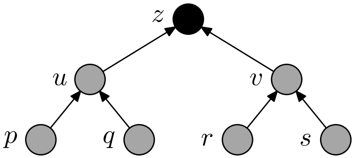


$$\left[\begin{array}{l} z_1 \\ \overline{z}_1 \end{array}\right]$$

Download axiom 8: $\overline{z}_1$

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target

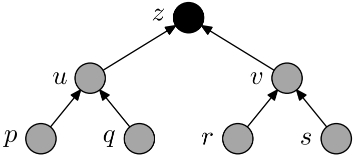


$$\begin{bmatrix} z_1 \\ \overline{z}_1 \end{bmatrix}$$

Infer 0 from $\overline{z}_1$ and z_1

Not True for $d = 1$ Variable per Vertex

1.	p_1	Source
2.	q_1	Source
3.	r_1	Source
4.	s_1	Source
5.	$\overline{p}_1 \vee \overline{q}_1 \vee u_1$	Pebbling
6.	$\overline{r}_1 \vee \overline{s}_1 \vee v_1$	Pebbling
7.	$\overline{u}_1 \vee \overline{v}_1 \vee z_1$	Pebbling
8.	$\overline{z}_1$	Target



$$\begin{bmatrix} z_1 \\ \overline{z}_1 \\ \color{red}{0} \end{bmatrix}$$

Infer 0 from
$\overline{z}_1$ and z_1

But Many Pebbles $\Rightarrow$ Many Clause for $d > 1$

This "top-down" proof in space 3 generalizes to any DAG G

- In terms of our induced pebble configurations:
  white pebbles are free for $d = 1$!
- In a sense, this is exactly why $Sp\left(Peb_G^1 \vdash 0\right) = \mathcal{O}(1)$
- But for $d > 1$ variables per vertex we can prove that
  # clauses $\geq$ # induced pebbles

Clauses $\geq$ # Induced Pebbles (Theorem)

F implies D minimally if $F \vDash D$ but $F' \nvDash D$ for all $F' \subsetneq F$.

Lemma

Suppose that F implies D minimally. For V any subset of variables, let $F_V = \{C \in F \mid Vars(C) \cap V \neq \emptyset\}$. Then for all $V \subseteq Vars(F) \setminus Vars(D)$ it holds that $|F_V| > |V|$.

Theorem

Suppose that $\mathbb{C}$ is a set of clauses derived from Peb^d_G for $d \geq 2$ and that $V \subseteq V(T)$ is a set of vertices such that $\mathbb{C}$ induces a black or white pebble on each $v \in V$. Then $|\mathbb{C}| \geq |V|$.

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Theorem

Suppose that $\mathbb{C}$ is a set of clauses derived from Peb^d_G for $d \geq 2$ and that $V \subseteq V(T)$ is a set of vertices such that $\mathbb{C}$ induces a black or white pebble on each $v \in V$. Then $|\mathbb{C}| \geq |V|$.

Clauses $\geq$ # Induced Pebbles (Informal Argument)

- If $\mathbb{C}$ induces N black pebbles and *no white pebbles* this is *almost* as if $\mathbb{C}$ implied N disjoint clauses
- If so, easy to believe that $|\mathbb{C}| \geq N$
- Problem: When $\mathbb{C}$ induces white pebbles on W, we get a bound not for $|\mathbb{C}|$ but for $|\mathbb{C} \cup \{\bigvee_{i=1}^{d} w_i \mid w \in W\}|$
- But every white pebble w contributes only 1 clause $\bigvee_{i=1}^{d} w_i$ but $d > 1$ new variables
- $\mathbb{C}$ must contain $d - 1$ more clauses for every white pebble to eliminate these new variables
- Formalizing this yields the stated lower bound on $|\mathbb{C}|$ in # induced pebbles.

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Induced Pebbles Break The Pebbling Rules

Unfortunately, our interpretation of $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ does not yield "well-behaved" pebbling $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$

- Erasures can (and will) lead to large blocks of black and white pebbles suddenly just disappearing
- Need to keep track of *exactly* which white pebbles have been used to get a black pebble on a vertex

"Illegal" removal of white pebble from *w* OK if all black pebbles above *w* dependent on this white pebble are removed as well!

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Pebble Subconfiguration $v\langle W\rangle$

Write $v\langle W\rangle$ to denote a pebble subconfiguration:

- black pebble on v together with
- white pebbles on W below v *thanks to which we have the black pebble on v*

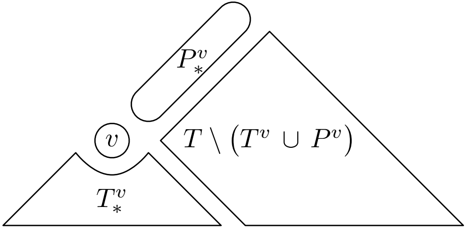
Redefine black-white pebble game in terms of moves with pebble subconfigurations instead of individual pebbles

Pebble Subconfiguration Terminology

For a pebble subconfiguration $v\langle W\rangle$:

- White pebbles on $w \in W$ support black pebble on v.
- Black pebble on v is dependent on white pebbles on W (for $W = \emptyset$ call $v\langle\emptyset\rangle$ an independent black pebble).
- A set of pebble subconfigurations $\mathbb{L}$ is a labelled pebble configuration or L-configuration.

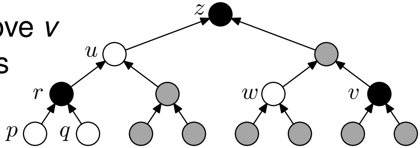
Notation for Vertex Sets in Binary Tree T



succ(v)	immediate successor of v ($\emptyset$ for root z)
pred(v)	immediate predecessors of v ($\emptyset$ for leaf)
T^v	vertices in the complete subtree of T rooted at v
T_*^v	T^v without its root, i.e., $T^v \setminus \{v\}$
P^v	vertices in the path from v to the root z of T
P_*^v	the path without v, i.e., $P^v \setminus \{v\}$

Order Relation on Pebble Subconfigurations

For $u\langle U\rangle$ and $v\langle V\rangle$, if u is above v and U is below V then $u\langle U\rangle$ is stronger than $v\langle V\rangle$



$$\mathbb{L} = \{z\langle u, v\rangle,\ r\langle p, q\rangle,\ w\langle \emptyset\rangle\}$$

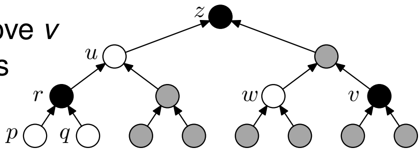
Formally:

- $v\langle V\rangle \preceq u\langle U\rangle$ if $T^v \setminus \bigcup_{w \in V} T^w \subseteq T^u \setminus \bigcup_{w \in U} T^w$
- $v\langle V\rangle \prec u\langle U\rangle$ if $v\langle V\rangle \preceq u\langle U\rangle$ and $v\langle V\rangle \neq u\langle U\rangle$

Note that $v\langle \emptyset\rangle \prec z\langle u, w\rangle$ in picture above

Order Relation on Pebble Subconfigurations

For $u\langle U\rangle$ and $v\langle V\rangle$, if u is above v and U is below V then $u\langle U\rangle$ is stronger than $v\langle V\rangle$



$\mathbb{L} = \{z\langle u, v\rangle, r\langle p, q\rangle, w\langle\emptyset\rangle\}$

Formally:

- $v\langle V\rangle \preceq u\langle U\rangle$ if $T^v \setminus \bigcup_{w\in V} T^w \subseteq T^u \setminus \bigcup_{w\in U} T^w$
- $v\langle V\rangle \prec u\langle U\rangle$ if $v\langle V\rangle \preceq u\langle U\rangle$ and $v\langle V\rangle \neq u\langle U\rangle$

Note that $v\langle\emptyset\rangle \prec z\langle u, w\rangle$ in picture above

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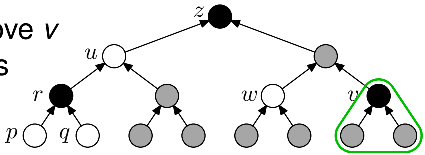

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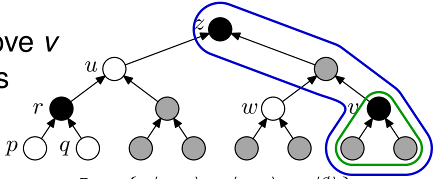

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Informal Description of Our Modified Pebble Game

- Pebble placement: always black on v together with whites on $pred(v)$, except for leaves where $pred(v) = \emptyset$
- Removal of white pebbles always allowed, but if so any black pebble dependent on these white pebbles must be removed as well
- "Traditional" removal of white pebble from w corresponds to merger of $v\langle V\rangle$ and $w\langle W\rangle$ into $v\langle (V \cup W) \setminus \{w\}\rangle$ followed by erasure of $v\langle V\rangle$ and $w\langle W\rangle$
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Definition (Labelled pebble game)

A labelled black-white pebbling, or L-pebbling, is a sequence $\mathcal{L} = \{\mathbb{L}_0, \ldots, \mathbb{L}_\tau\}$ of L-configurations $\mathbb{L}_t$ such that $\mathbb{L}_0 = \{\emptyset\}$ and $\mathbb{L}_t$ is obtained from $\mathbb{L}_{t-1}$ by one of the following:

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Pebbling Price in Labelled Pebble Game

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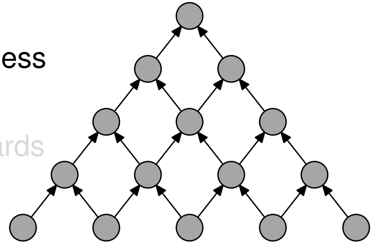

- White pebbles may slide upwards and black pebbles may slide downwards
- Destroys pebbling price for general graphs
- But still pebbling price $\Omega(h)$ for binary trees T_h

Theorem

For T_h a complete binary tree of height h,
$L\text{-}Peb(T_h) = \Theta(BW\text{-}Peb(T_h)) = \Theta(h)$.

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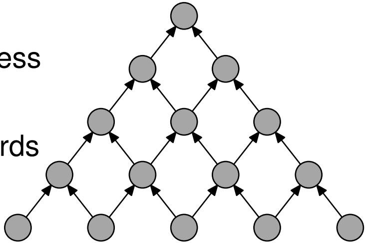

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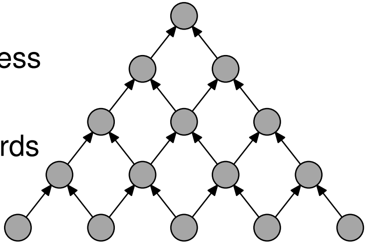

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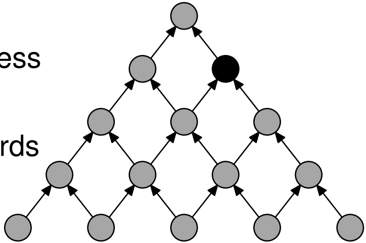

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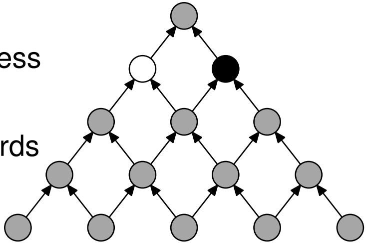

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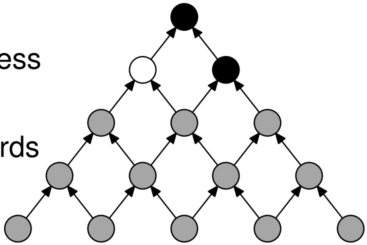

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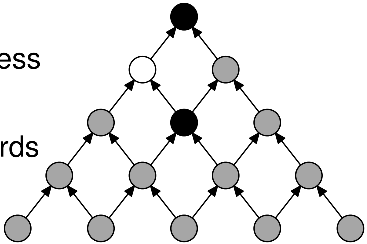

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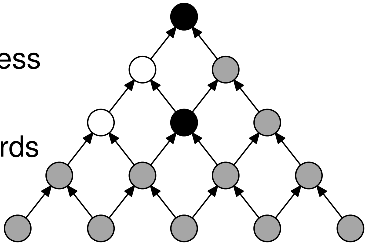

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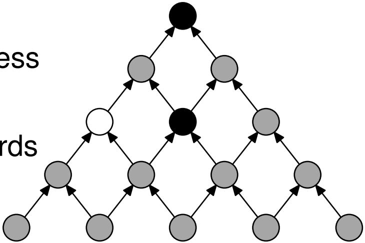

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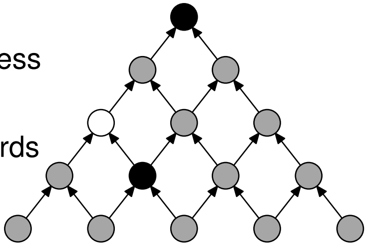

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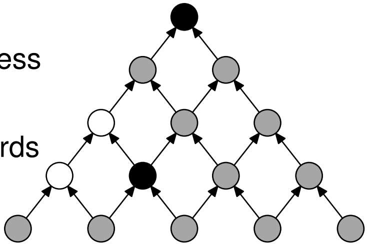

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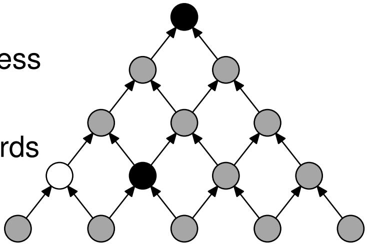

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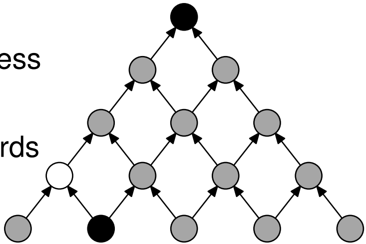

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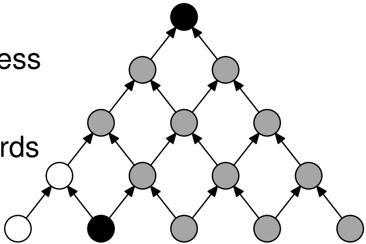

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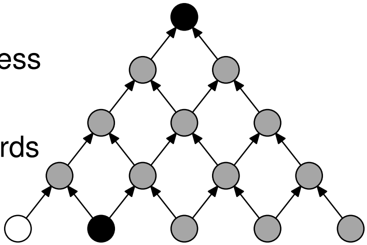

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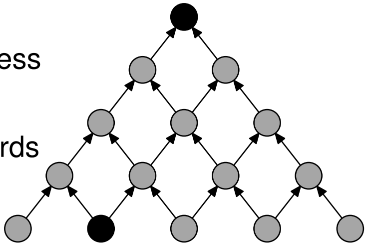

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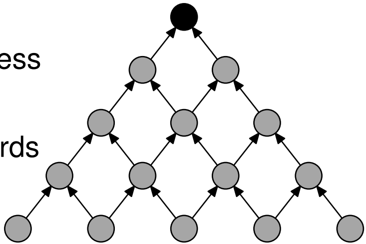

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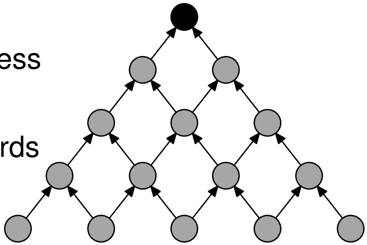

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$\textit{L-Peb}(T_h) = \Theta\big(\textit{BW-Peb}(T_h)\big) = \Theta(h)$.

Formal Definition of Induced Pebble Subconfigurations

Formal definition of induced pebbles turns out quite involved:

Definition (Induced L-configuration)

If for a vertex v there is a *minimal* set $W' \subseteq T \setminus P^v$ such that

- $\mathbb{C} \cup \left\{ \bigvee_{i=1}^{d} w_i \mid w \in W' \right\} \vDash \bigvee_{u \in P^v} \bigvee_{i=1}^{d} u_i$ but
- $\mathbb{C} \cup \left\{ \bigvee_{i=1}^{d} w_i \mid w \in W' \right\} \nvDash \bigvee_{u \in P^v_*} \bigvee_{i=1}^{d} u_i$

then $\mathbb{C}$ induces the pebble subconfiguration $v\langle W \rangle$ for

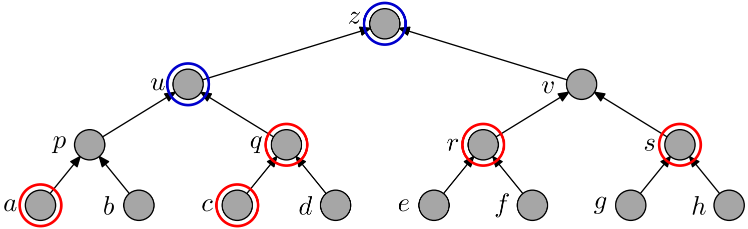
$$W = \left\{ w \in W' \cap T^v_* \mid P^w_* \cap W' = \emptyset \right\}.$$

The induced L-configuration $\mathbb{L}(\mathbb{C})$ of a clause configuration $\mathbb{C}$ consists of all such induced pebble subconfigurations $v\langle W \rangle$.

Example Induced Pebble Subconfiguration

For $d = 1$, the clause configuration

$$\mathbb{C} = [\, \overline{a}_1 \lor \overline{c}_1 \lor \overline{q}_1 \lor \overline{r}_1 \lor \overline{s}_1 \lor u_1 \lor z_1 \,]$$



induces the pebble subconfiguration $u\langle a, q\rangle$

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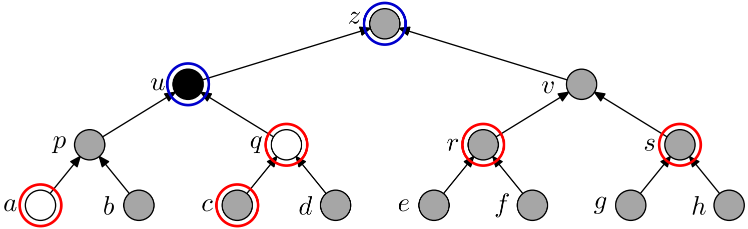

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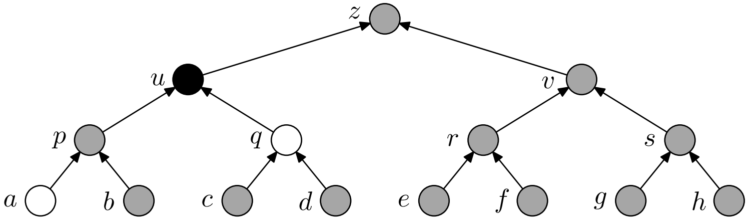

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Resolution Derivations Induce Labelled Pebblings

Technical detail: study derivations of $\bigvee_{i=1}^{d} z_i$ from $*Peb_T^d = Peb_T^d \setminus \{\overline{z}_1, \ldots, \overline{z}_d\}$ instead

(Same space $Sp(Peb_{T_h}^d \vdash 0) = Sp(*Peb_{T_h}^d \vdash \bigvee_{i=1}^{d} z_i)$ anyway)

Theorem

Let $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ be a derivation of $\bigvee_{i=1}^{d} z_i$ from $*Peb_T^d$.

Then the induced L-configurations $\{\mathbb{L}(\mathbb{C}_0), \ldots, \mathbb{L}(\mathbb{C}_\tau)\}$ form the "backbone" of an L-pebbling $\mathcal{L}$ of T in the following sense:

All transitions $\mathbb{L}(\mathbb{C}_t) \rightsquigarrow \mathbb{L}(\mathbb{C}_{t+1})$ can be done with L-pebbling moves in such a way that $cost(\mathcal{L}) = \mathcal{O}(\max_{t\in[\tau]}\{cost(\mathbb{L}(\mathbb{C}_t))\})$.

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Main Theorem

Theorem

The space of refuting the pebbling contradiction of degree $d \geq 2$ over the complete binary tree of height h in resolution is $Sp\left(Peb^d_{T_h} \vdash 0\right) = \Theta(h)$.

Proof sketch.

- Upper bound easy (use "black-pebbling" resolution proof)
- For lower bound, let $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ be derivation of $\bigvee_{i=1}^{d} z_i$ from $*Peb^d_{T_h}$ in minimal space $= Sp\left(Peb^d_{T_h} \vdash 0\right)$
- Then there is some $\mathbb{C}_t \in \pi$ that induces $\Omega(h)$ pebbles in T_h
- Thus $Sp(\pi) \geq |\mathbb{C}_t| \geq$ # pebbles induced by $\mathbb{C}_t = \Omega(h)$. □

A Separation of Space and Width in Resolution

Corollary

For all $k \geq 4$, there is a family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with refutation width $W(F_n \vdash 0) = \mathcal{O}(1)$ and refutation space $Sp(F_n \vdash 0) = \Theta(\log n)$.

Proof.

We know $W\big(Peb_G^d \vdash 0\big) = \mathcal{O}(d)$ for all G.

Fix $d \geq 2$, let $F_n = Peb_{T_h}^d$ for $h = \lfloor \log(n+1) \rfloor$ and use the Main Theorem. □

Conclusion

- First lower bound on space in resolution which is not the consequence of a lower bound on width but instead separates the two measures
- Answers an open question in several previous papers
- We believe that it should be possible to strengthen this result in (at least) two ways

Open Problems

Extend to arbitrary DAGs

Conjecture 1

For G an arbitrary DAG with a unique target and with all vertices having indegree 0 or 2, if $d \geq 2$ it holds that $Sp\big(Peb_G^d \vdash 0\big) = \Omega\big(BW\text{-}Peb(G)\big)$.

Would yield almost optimal separation $\Omega(n/\log n)$ between space and width

Best conceivable is $\Omega(n)$

Open Problems (cont.)

Generalize to k-DNF resolution and prove space hierarchy

k-DNF resolution: lines in proof not disjunctive clauses but disjunctions of conjunctions of size $\leq k$

Conjecture 2

For k-DNF resolution refutations of pebbling contradictions defined over complete binary trees T_h of height h, fixing k it holds that $Sp_{\mathfrak{Res}(k+1)}\left(Peb_{T_h}^{k+1} \vdash 0\right) = \mathcal{O}(1)$ but $Sp_{\mathfrak{Res}(k)}\left(Peb_{T_h}^{k+1} \vdash 0\right) = \Omega(h)$.

Would show that k-DNF resolution proof systems for increasing k form strict space hierarchy

Related Question for *k*-DNF Resolution

- For minimally unsatisfiable CNF formulas it is well-known that # clauses $>$ # variables
- What about size of minimally unsatisfiable sets of *k*-DNF formulas in terms of number of variables?
- Good lower bound would probably be enough to prove space hierarchy for *k*-DNF resolution

References

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Extended abstract in proceedings of 38th ACM Symposium on Theory of Computing (STOC '06)

That’s basically it!

(modulo all the gory technical details)