Towards an Optimal Separation of Space and Length in Resolution

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Outline

1. A Proof Complexity Primer
   - Some Background
   - Definitions and Notation
   - Highlights of Research Results
2. Our Contribution: Lower Bounds on Space
   - Pebble Games
   - Pebbling Contradictions
   - Outline of Proofs
3. Some Open Problems
   - A List of Some Nice Open Problems
   - Two Possible Lines of Attack for the Nicest Problem

A Fundamental Problem in Computer Science

Problem

Given a propositional logic formula F, is it true no matter how we assign values to its variables?

TAUTOLOGY: Fundamental problem in Theoretical Computer Science since Cook's NP-completeness paper (1971)

Last decade or so: also intense applied interest

Enormous progress on algorithms (although still exponential time in worst case)

Proof Complexity

Proof search algorithm: proof system with derivation rules

Proof complexity: study of proofs in such systems

- Lower bounds: no algorithm can do better (even optimal one always guessing the right move)
- Upper bounds: gives hope for good algorithms if we can search for proofs in system efficiently

Resolution

- Prove tautologies ⇔ refute unsatisfiable formulas in conjunctive normal form (CNF)
- Resolution: proof system for refuting CNF formulas
- Perhaps *the* most studied system in proof complexity
- Also used in many real-world automated theorem provers
- Basis of current state-of-the-art algorithms (winners in SAT 2007 competition: resolution + clause learning)

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \ldots \lor a_k$: disjunction of literals
  At most k literals: k-clause
- CNF formula $F = C_1 \land \ldots \land C_m$: conjunction of clauses
  k-CNF formula: CNF formula consisting of k-clauses
  (assume k fixed)
- Refer to clauses of CNF formula as axioms
  (as opposed to derived clauses)
- $F \vDash C$: semantical implication, $\alpha(F)$ true $\Rightarrow \alpha(C)$ true
  for all truth value assignments α

Resolution Rule

Resolution rule:

$$\frac{B \vee x \quad C \vee \overline{x}}{B \vee C}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $C_1, C_2 \in F$ by the resolution rule, then $F \wedge D$ is satisfiable.

Prove F unsatisfiable by deriving the unsatisfiable empty clause 0 (the clause with no literals) from F by resolution

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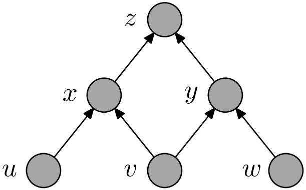
Prove F unsatisfiable by deriving the unsatisfiable empty clause 0 (the clause with no literals) from F by resolution

Example CNF Formula

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

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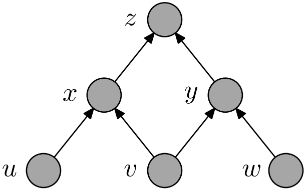

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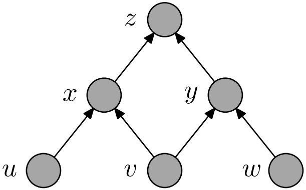

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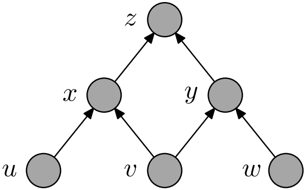

Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	0
# literals in largest clause	0
# lines on blackboard used	0

Can write down axioms,
erase used clauses or
infer new clauses (but only from clauses currently on the board!)

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	1
# literals in largest clause	1
# lines on blackboard used	1

Blackboard:

u

Write down axiom 1: u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	2
# literals in largest clause	1
# lines on blackboard used	2

Blackboard:

u
v

Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	3
# literals in largest clause	3
# lines on blackboard used	3

u
v
$\overline{u} \lor \overline{v} \lor x$

Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

u
v
$\overline{u} \vee \overline{v} \vee x$
$\overline{v} \vee x$

Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Erase clause $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

u
v
$\overline{v} \lor x$

Erase clause $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

u
v
$\overline{v} \lor x$

Erase clause u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

v
$\overline{v} \lor x$

Erase clause u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	4
# literals in largest clause	3
# lines on blackboard used	4

v
$\overline{v} \lor x$

Infer x from
v and $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

v
$\overline{v} \lor x$
x

Infer x from v and $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

v
$\overline{v} \lor x$
x

Erase clause $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

v

x

Erase clause $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

v
x

Erase clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	5
# literals in largest clause	3
# lines on blackboard used	4

x

Erase clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	6
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{x} \lor \overline{y} \lor z$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	6
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{x} \lor \overline{y} \lor z$

Infer $\overline{y} \lor z$ from x and $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Infer $\overline{y} \lor z$ from x and $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase clause $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{y} \lor z$

Erase clause $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

x
$\overline{y} \vee z$

Erase clause x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	7
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$

Erase clause x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	8
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Infer $\overline{v} \lor \overline{w} \lor z$ from $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee z$

Erase clause $\overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \vee \overline{w} \vee z$

Erase clause $\overline{y} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	9
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \lor \overline{w} \lor z$
v

Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	10
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \lor \overline{w} \lor z$
v
w

Write down axiom 3: w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	11
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$

Write down axiom 7: $\overline{z}$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	11
# literals in largest clause	3
# lines on blackboard used	4

$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$

Infer $\overline{w} \vee z$ from v and $\overline{v} \vee \overline{w} \vee z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
# literals in largest clause	3
# lines on blackboard used	5

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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Blackboard bookkeeping	
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w
$\overline{z}$
$\overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
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7. $\overline{z}$

Blackboard bookkeeping	
# distinct clauses on board	12
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w
$\overline{z}$
$\overline{w} \vee z$

Infer z from
w and $\overline{w} \vee z$

Example Resolution Refutation

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2. v
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4. $\overline{u} \lor \overline{v} \lor x$
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Erase clause $\overline{w} \lor z$

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$\overline{z}$
z

Infer 0 from
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Blackboard bookkeeping	
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$\overline{z}$
z
0

Infer 0 from $\overline{z}$ and z

Length, Width and Space

- Length $L(\pi)$ of refutation $\pi : F \vdash 0$
  # distinct clauses in all of π
  (in our example 14)

- Width $W(\pi)$ of refutation $\pi : F \vdash 0$
  # literals in largest clause in π
  (in our example 3)

- Space $Sp(\pi)$ of refutation $\pi : F \vdash 0$
  max # clauses on blackboard simultaneously
  (in our example 5)

Length, Width and Space of Refuting F

- Length of refuting F is

$$L(F \vdash 0) = \min_{\pi : F \vdash 0} \{ L(\pi) \}$$

- Width of refuting F is

$$W(F \vdash 0) = \min_{\pi : F \vdash 0} \{ W(\pi) \}$$

- Space of refuting F is

$$Sp(F \vdash 0) = \min_{\pi : F \vdash 0} \{ Sp(\pi) \}$$

Why Should We Care About These Measures?

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm
- Width: Intimately connected to length and space ☺

Can also give ideas for proof search heuristics

When comparing measures, for simplicity consider
k-CNF formulas (during this talk)

Results for Length

Easy upper bound: $L(F \vdash 0) \leq 2^{(\#\text{ variables in } F + 1)}$

Theorem (Haken 1985)

Polynomial-size CNF formula family with (weakly) exponential lower bound on refutation length (pigeonhole principle)

Later improved to truly exponential lower bounds for different formula families (Urquhart 1987, Chvátal & Szemerédi 1988 and others)

But resolution used widely in practice anyway
Amenable to proof search because of its simplicity

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Connection Between Length and Width (1/2)

Trivial upper bound: $W(F \vdash 0) \leq \#$ variables in F

Also, a narrow resolution refutation is necessarily short

For a refutation in width w, bound on length $\leq (2 \cdot \#\text{ variables})^w$ (max # distinct clauses)

Connection Between Length and Width (2/2)

There is a kind of converse to this:

Theorem (Ben-Sasson & Wigderson 1999)

The width of refuting a k-CNF formula F over n variables is

$$W(F \vdash 0) = \mathcal{O}\left(\sqrt{n \log L(F \vdash 0)}\right).$$

Proof search heuristic: search for narrow refutations!

Two comments:

- Short and narrow refutation need not be the same one!?
- Bound on width in terms of length essentially optimal (Bonet & Galesi 1999)

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Results for Space

- Easy upper bound: $Sp(F \vdash 0) \leq$ size of F, or more precisely $\leq \min(\#\text{ variables in } F, \#\text{ clauses in } F) + \mathcal{O}(1)$
- Many lower bounds proven, e.g. for polynomial-size k-CNF formula families matching upper bounds above (Torán 1999, Alekhnovich et al. 2000)
- Also, all space lower bounds turned out to match width lower bounds! True in general?

Connection Between Space and Width

Theorem (Atserias & Dalmau 2003)

For any unsatisfiable k-CNF formula F it holds that

$$Sp(F \vdash 0) \geq W(F \vdash 0) - \mathcal{O}(1).$$

But do space and width always coincide?
Are they in fact the same measure asymptotically?

Or can they be separated?
I.e., is there a k-CNF formula family $\{F_n\}_{n=1}^{\infty}$ such that $Sp(F_n \vdash 0) = \omega(W(F_n \vdash 0))$?

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Separation of Space and Width

Theorem (Nordström 2006)

For all $k \geq 4$, there is a family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with

- *refutation length* $L(F_n \vdash 0) = \mathcal{O}(n)$,
- *refutation width* $W(F_n \vdash 0) = \mathcal{O}(1)$ *and*
- *refutation space* $Sp(F_n \vdash 0) = \Theta(\log n)$.

So space and width are not "the same"

But very weak separation—not end of story?

Connection Between Space and Length?

Length and width tightly related:
$\exists$ short refutations $\Leftrightarrow$ $\exists$ (resonably) narrow refutations

What about length v.s. space?

- Small space $\Rightarrow$ short length (easy)
- But does short length imply small space?
- Or are there formulas with short, easy refutations that must require large space?

Mentioned as open problem in several papers
Apparently no consensus on what the "right answer" should be

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Towards an Optimal Separation of Space and Length

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Best separation of space and length so far + exponential improvement of previous space-width separation

Indicates that "right answer" should be optimal separation of space and length with length $\mathcal{O}(n)$ and space $\Omega(n/\log n)$

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Any Practical Implications?

Yes and no

Space measures memory consumption of clause learning algorithms, but is "wrong measure" for practical purposes

Always space $\leq$ formula size, but practical applications usually will have much more memory available than that

But maybe lower bounds on space can give clue about hardness anyway

(Sabharwal et al. 2003) exhibits formulas with very short refutations that state-of-the-art SAT-solver cannot find

Exactly the formulas in our $\Theta(\sqrt{n})$ space bound!

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How to Separate Length and Space?

Want to find formulas that

- can be quickly refuted
- but require large space

Such time-space trade-off questions well-studied for pebble games modelling calculations described by DAGs

- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

Known result: ∃ DAGs requiring many pebbles in terms of size

Look at CNF formulas encoding pebbles games on DAGs!

The Black-White Pebble Game

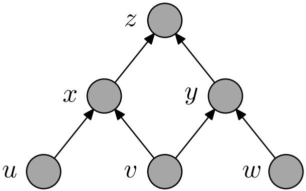
Start with all vertices of DAG G empty

1. Can place black pebble on (empty) vertex v if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble from v if all immediate predecessors have pebbles on them

Goal: get black pebble on sink vertex of G with no other pebbles in G, using as few pebbles as possible

Studied by Cook & Sethi (1976) and many others

Example Pebbling and Pebbling Price



- Cost of pebbling:
  max # pebbles simultaneously in G
  (in our example 4)
- Black-white pebbling price *BW-Peb*(G) of DAG G:
  minimal cost of any pebbling
- (Black) pebbling price *Peb*(G):
  minimal cost of pebbling using black pebbles only

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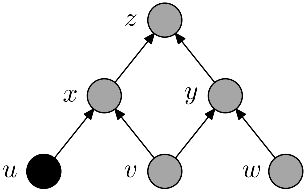

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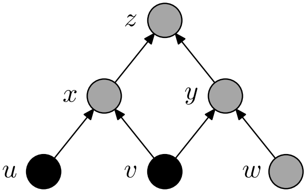

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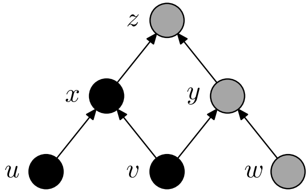

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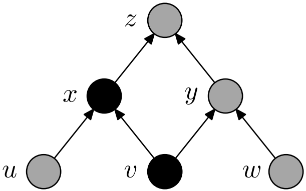

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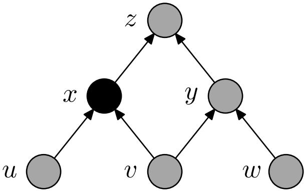

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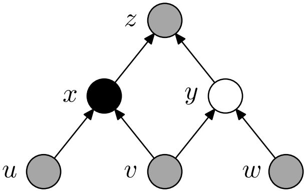

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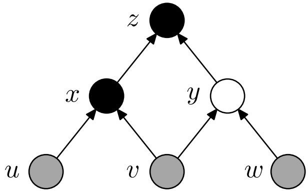

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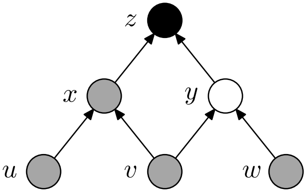

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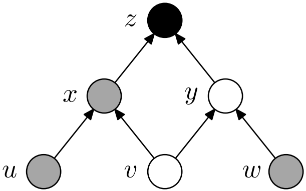

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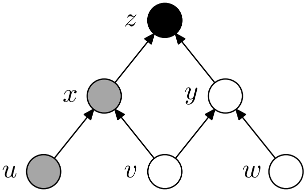

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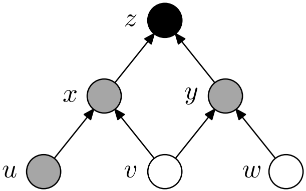

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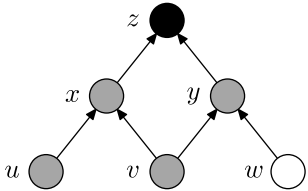

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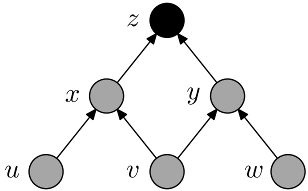

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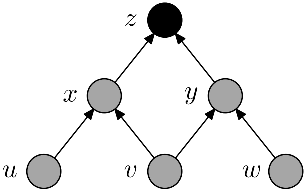

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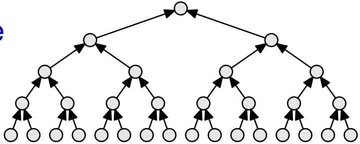
Pebbling Price of Binary Trees

Let T_h denote complete binary tree of height h considered as a DAG



- Pebbling price of T_h is

$$Peb(T_h) = h + 2$$

  (easy induction over the tree height)

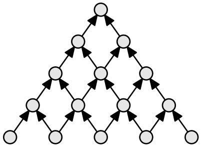
- Black-white pebbling price is

$$BW\text{-}Peb(T_h) = \left\lfloor \frac{h}{2} \right\rfloor + 3 = \Omega(h)$$

  (Lengauer & Tarjan 1980)

Pebbling Price of Pyramids

Let Π_h denote pyramid graph of height h considered as a DAG



- $Peb(\Pi_h) = h + 2$
  (Cook 1974)
- $BW\text{-}Peb(\Pi_h) = \left\lfloor \frac{h}{2} \right\rfloor + \mathcal{O}(1) = \Omega(h)$
  (Klawe 1985)

DAG Size-Pebbling Price Trade-off

- Binary tree of size n has pebbling price $\Theta(\log n)$
- Pyramid of size n has pebbling price $\Theta\left(\sqrt{n}\right)$

Pebbling Contradiction

CNF formula encoding pebble game on DAG G with unique sink z and all non-source vertices having indegree 2

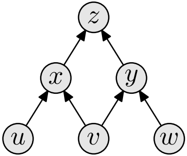
Associate d variables $v_1, \ldots, v_d$ with every vertex $v \in V(G)$

The dth degree pebbling contradiction Peb^d_G over G says that:

- All source vertices have at least one true variable
- Truth propagates upwards according to pebbling rules
- For the sink z all variables are false

Studied by Bonet et al. (1998), Raz & McKenzie (1999), Ben-Sasson & Wigderson (1999) and others

Pebbling Contradiction $Peb^2_{\Pi_2}$ for Pyramid of Height 2



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\begin{aligned}
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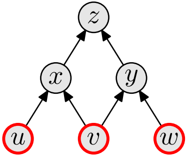

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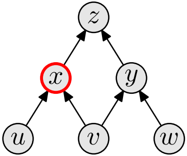

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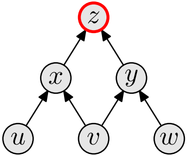

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Pebbling Contradictions Easy w.r.t. Length and Width

Peb^d_G is an unsatisfiable (2+d)-CNF formula with

- $d \cdot |V(G)|$ variables
- $\mathcal{O}\left(d^2 \cdot |V(G)|\right)$ clauses

Can be refuted by deriving $\bigvee_{i=1}^{d} v_i$ for all $v \in V(G)$ inductively in topological order and resolving with sink axioms $\overline{z}_i$, $i \in [d]$

It follows that

- $L(F \vdash 0) = \mathcal{O}\left(d^2 \cdot |V(G)|\right)$
- $W(F \vdash 0) = \mathcal{O}(d)$

(Ben-Sasson et al. 2000)

What about Pebbling Contradictions and Space?

Upper bounds:

- Arbitrary DAGs G
  optimal black pebbling of G + proof from previous slide:
  $Sp(Peb_G^d \vdash 0) \leq Peb(G) + \mathcal{O}(1)$
- Binary trees T_h
  improvement by Esteban & Torán (2003):
  $Sp(Peb_{T_h}^2 \vdash 0) \leq \frac{2}{3} Peb(T_h) + \mathcal{O}(1)$
- Only one variable / vertex
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No lower bounds for $d \geq 2$ known previous to our work

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Rephrasing Our Results

Theorem (Nordström 2006)

The space of refuting pebbling contradictions $Peb^d_{T_h}$ of degree $d \geq 2$ over binary trees of height h is $Sp\left(Peb^d_{T_h} \vdash 0\right) = \Theta(h)$.

Theorem (Nordström & Håstad 2008)

The space of refuting pebbling contradictions $Peb^d_{\Pi_h}$ of degree $d \geq 2$ over pyramids of height h is $Sp\left(Peb^d_{\Pi_h} \vdash 0\right) = \Theta(h)$.

Previously stated theorems follow as corollaries since

- height $=$ log(tree size)
- height $= \sqrt{\text{pyramid size}}$

Proof Idea

Prove lower bounds on space of $\pi : Peb^d_G \vdash 0$ by

1. Interpreting sets of clauses $\mathbb{C}$ in terms of black and white pebbles on G
2. Showing that if $\mathbb{C}$ corresponds to N pebbles it contains at least N clauses (if $d \geq 2$)
3. Establishing that resolution refutations induce black-white pebblings under this interpretation

Then some $\mathbb{C} \in \pi$ must induce $BW\text{-}Peb(G)$ pebbles

$$\Downarrow$$

$$|\mathbb{C}| \geq BW\text{-}Peb(G)$$

$$\Downarrow$$

$$Sp(Peb^d_G \vdash 0) = \Omega(BW\text{-}Peb(G))$$

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Interpreting Clauses in Terms of Pebbles

Black-white pebbling models non-deterministic computation

- black pebbles $\Leftrightarrow$ known results
- white pebbles $\Leftrightarrow$ assumptions needing to be verified

Want to translate a set of clauses $\mathbb{C}$ into black and white pebbles using this intuition

Consider the semantic content of $\mathbb{C}$, i.e., what clauses it implies

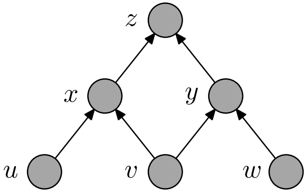
Intuition for Black Pebbles

Induced Black Pebble

Fruitful to think of black pebble on v as truth of v
I.e., place a black pebble on v if $\mathbb{C} \vDash \bigvee_{i=1}^{d} v_i$

Propagation of truth similar to rules for black pebbling

Consider (fast-forward version of) our resolution refutation example again:



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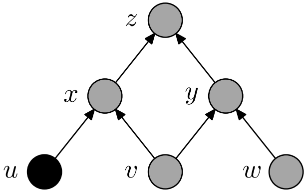

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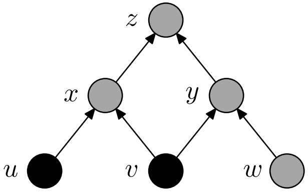

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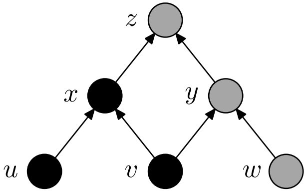

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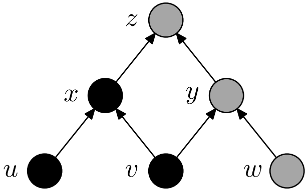

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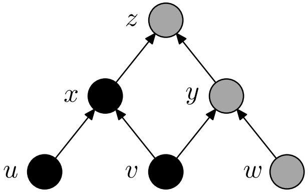

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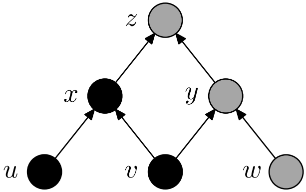

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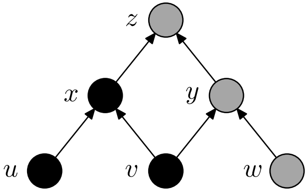

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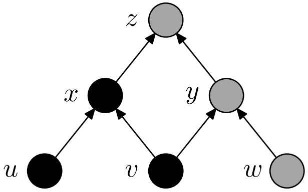

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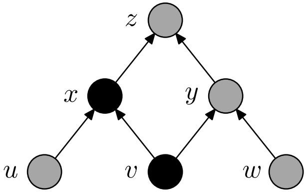

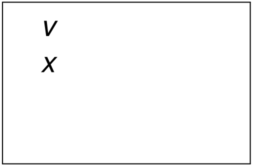

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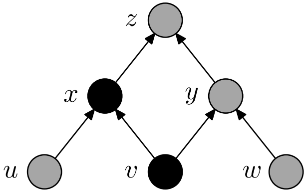

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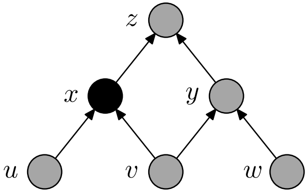

Intuition for White Pebbles

White pebbles are slightly trickier to get a handle on

“We know z given v, w”

Corresponds to that $\mathbb{C}$ implies $\bigvee_{l=1}^{d} z_l$ if we also assume $\bigvee_{i=1}^{d} v_i, \bigvee_{j=1}^{d} w_j$

This is the case for $\mathbb{C} = \{\overline{v} \lor \overline{w} \lor z\}$ in our example refutation

Induced White Pebbles

$\mathbb{C}$ should induce white pebbles on set of vertices W if $\mathbb{C} \cup \{\bigvee_{i=1}^{d} w_i \mid w \in W\} \vDash \bigvee_{i=1}^{d} v_i$

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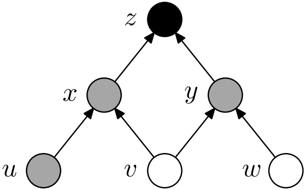

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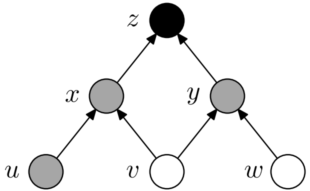

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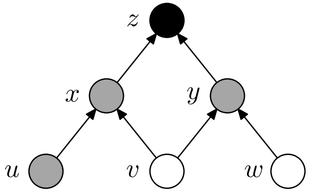
$\mathbb{C}$ should induce white pebbles on set of vertices W if $\mathbb{C} \cup \{\bigvee_{i=1}^{d} w_i \mid w \in W\} \vDash \bigvee_{i=1}^{d} v_i$

Intuition for White Pebbles

White pebbles are slightly trickier to get a handle on



"We know z given v, w"

Corresponds to that $\mathbb{C}$ implies $\bigvee_{l=1}^{d} z_l$ if we also assume $\bigvee_{i=1}^{d} v_i, \bigvee_{j=1}^{d} w_j$

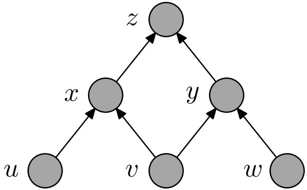
This is the case for $\mathbb{C} = \{\overline{v} \vee \overline{w} \vee z\}$ in our example refutation

Induced White Pebbles

$\mathbb{C}$ should induce white pebbles on set of vertices W if $\mathbb{C} \cup \{\bigvee_{i=1}^{d} w_i \mid w \in W\} \vDash \bigvee_{i=1}^{d} v_i$

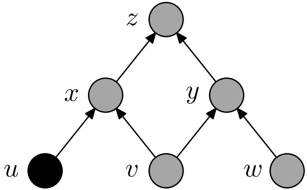
Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

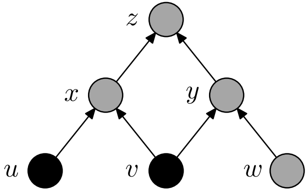


u

Write down axiom 1: u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

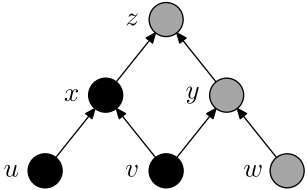


u
v

Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

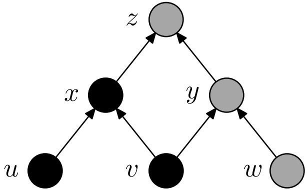


u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

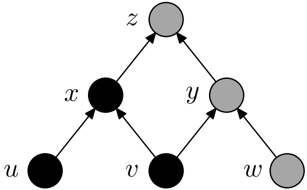


u
v
$\overline{u} \lor \overline{v} \lor x$

Infer $\overline{v} \lor x$ from u and $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

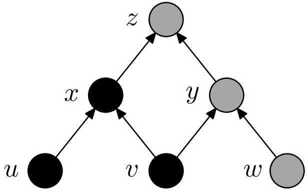


u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

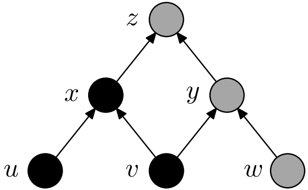


u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Erase clause $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

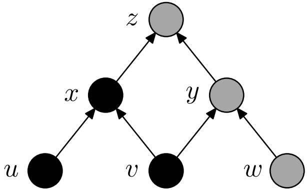


u
v
$\overline{v} \lor x$

Erase clause $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

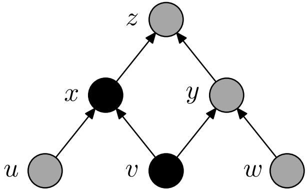


u
v
$\overline{v} \vee x$

Erase clause u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

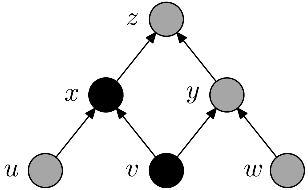


v
$\overline{v} \vee x$

Erase clause u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

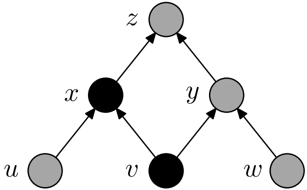


v
$\overline{v} \lor x$

Infer x from v and $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

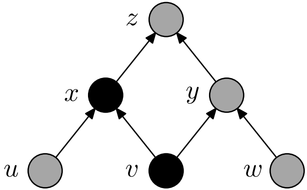


v
$\overline{v} \vee x$
x

Infer x from v and $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

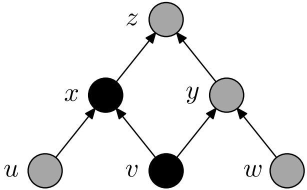


v
$\overline{v} \lor x$
x

Erase clause $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

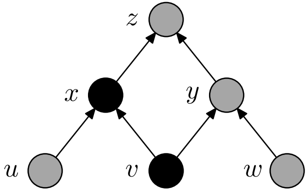


v
x

Erase clause $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



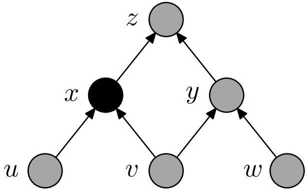
v

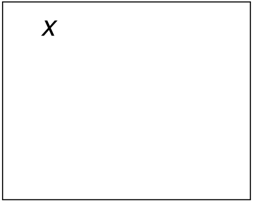
x

Erase clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

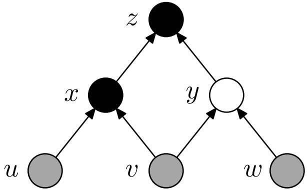




Erase clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

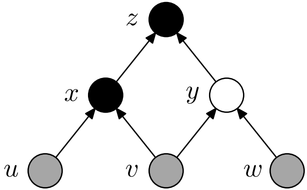


x
$\overline{x} \lor \overline{y} \lor z$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

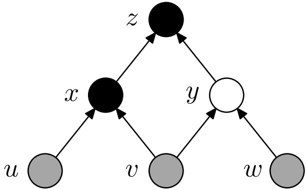


x
$\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

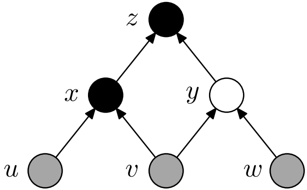


x
$\overline{x} \vee \overline{y} \vee z$
$\overline{y} \vee z$

Infer $\overline{y} \vee z$ from x and $\overline{x} \vee \overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

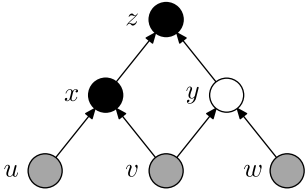


x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase clause $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

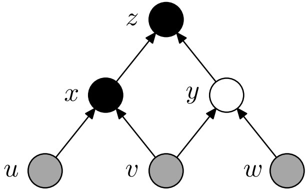


x
$\overline{y} \vee z$

Erase clause $\overline{x} \vee \overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

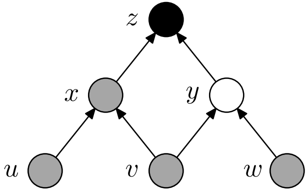


x
$\overline{y} \lor z$

Erase clause x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

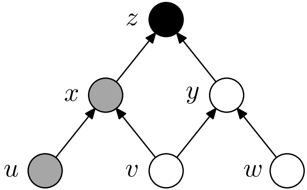


$\overline{y} \lor z$

Erase clause x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

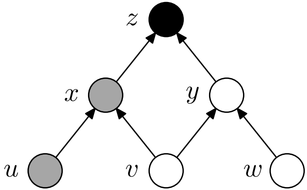


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

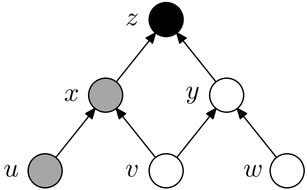


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

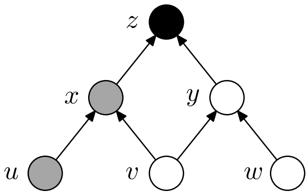


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Infer $\overline{v} \lor \overline{w} \lor z$ from $\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

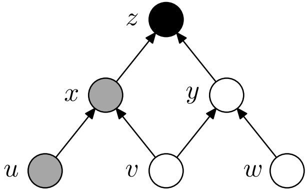


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

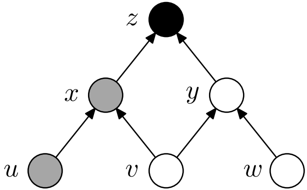


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

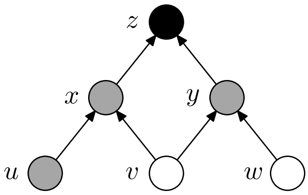


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee z$

Erase clause $\overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

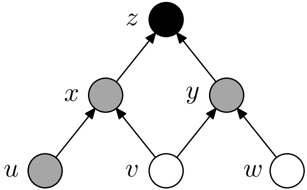


$\overline{v} \lor \overline{w} \lor z$

Erase clause $\overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

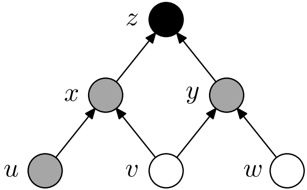


$\overline{v} \vee \overline{w} \vee z$
v

Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

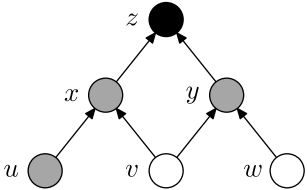


$\overline{v} \lor \overline{w} \lor z$
v
w

Write down axiom 3: w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$

Write down axiom 7: $\overline{z}$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
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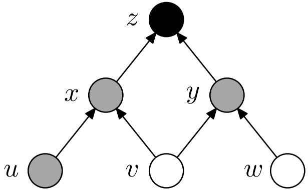

$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$

Infer $\overline{w} \vee z$ from v and $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
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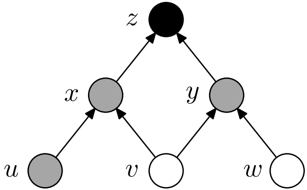

$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$
$\overline{w} \vee z$

Infer $\overline{w} \vee z$ from v and $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
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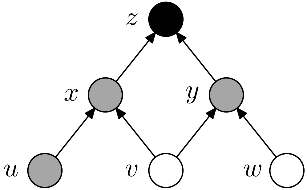

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
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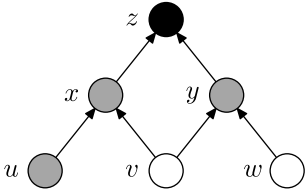

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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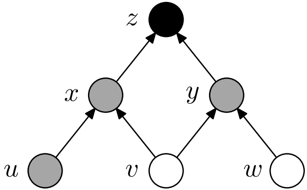

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Erase clause $\overline{v} \lor \overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
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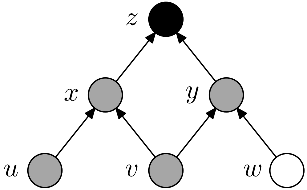

w
$\overline{z}$
$\overline{w} \vee z$

Erase clause $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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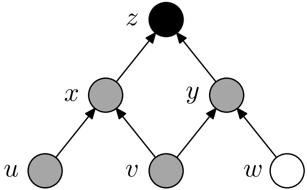

w
$\overline{z}$
$\overline{w} \lor z$

Infer z from
w and $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

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2. v
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4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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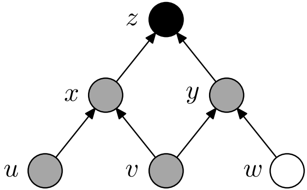

w
$\overline{z}$
$\overline{w} \lor z$
z

Infer z from
w and $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

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4. $\overline{u} \vee \overline{v} \vee x$
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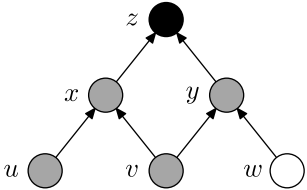

w
$\overline{z}$
$\overline{w} \vee z$
z

Erase clause w

Example of Refutation-Pebbling Correspondence

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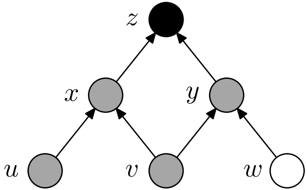

$\overline{z}$
$\overline{w} \vee z$
z

Erase clause w

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2. v
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4. $\overline{u} \lor \overline{v} \lor x$
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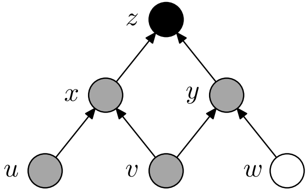

$\overline{z}$
$\overline{w} \lor z$
z

Erase clause $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
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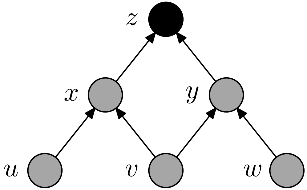

$\overline{z}$

z

Erase clause $\overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
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4. $\overline{u} \lor \overline{v} \lor x$
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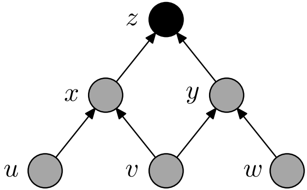

$\overline{z}$
z

Infer 0 from
$\overline{z}$ and z

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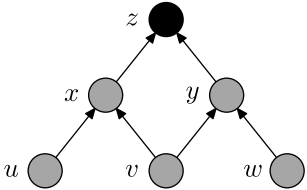

$\overline{z}$
z
0

Infer 0 from $\overline{z}$ and z

Sweeping the details under the rug...

This looks very nice, but in reality things get (much) messier

Refutations have no reason to derive nicely structured clauses
$\Rightarrow$ not possible to extract pebblings from refutations

Instead we invent new, modified pebble games and show:

1. Refutations induce pebblings in these modified games
2. Space is lower-bounded by modified pebbling price
3. Modified pebbling price asymptotically equals black-white pebbling price (currently only for trees and pyramids)

In this way get lower bound on space in terms of tree/pyramid height, which yields previously stated separations as corollaries

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Length-Width Trade-offs: Are Short Proofs Narrow?

Ben-Sasson & Wigderson (1999) showed that given refutation in length L, can find refutation in width $\mathcal{O}\left(\sqrt{n \log L}\right)$

But not the same refutation! Exponential blow-up in length!

Is this increase in length necessary?

Open Question (Informal)

Suppose that a k-CNF formula F has a short refutation. Does it then have a refutation that is both short and narrow?

Or are there formula families exhibiting length-width trade-off?

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Given refutation in small space $\Rightarrow$ exists refutation in short length (by Atserias & Dalmau 2003)

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Suppose that a k-CNF formula F has a refutation in small space. Does it then have a refutation in both short length and small space?

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Recall: $\exists$ short refutation $\Rightarrow$ $\exists$ (reasonably) narrow refutation

Is it true that $\exists$ short refutation $\Rightarrow$ $\exists$ small space refutation?

Or can short refutations be "arbitrarily complex" w.r.t. space?

My Conjecture

Exists family of k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ such that $L(F_n \vdash 0) = \mathcal{O}(n)$ but $Sp(F_n \vdash 0) = \Omega(n/\log n)$

Would separate length and space in strongest sense possible (given length n, space $\mathcal{O}(n/\log n)$ always possible)

Could be bad news for proof search algorithms

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Plausible Candidate: Pebbling Contradictions (Again)

Pebbling contradictions are refutable in linear length

For binary trees and pyramids, space grows like $\mathit{BW}\text{-}\mathit{Peb}(G)$

Intuition

For any DAG G, from resolution refutation of pebbling contradiction should be possible to extract black-white pebbling of underlying DAG

This would be sufficient!

There exists a family of DAGs $\{G_n\}_{n=1}^{\infty}$ of size $\mathcal{O}(n)$ with $\mathit{BW}\text{-}\mathit{Peb}(G_n) = \Theta(n / \log n)$ (Gilbert & Tarjan 1978)

Plausible Candidate: Pebbling Contradictions (Again)

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Two Possible Lines of Attack

There are (at least) two obvious ways of attacking this problem:

1. Prove that the modified pebbling price and the standard black-white pebbling price coincide for any DAG
2. Prove a lower bound on modified pebbling price for the Gilbert-Tarjan graphs

We are currently working on this…

References

Space-width separation published as *Narrow Proofs May Be Spacious: Separating Space and Width in Resolution*

- Extended abstract in STOC '06
- Journal version to appear in SIAM Journal on Computing

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Take-Home Message

- Lots of nice (and surprising!) results relating length, width and space
- Quite a few nice open problems left
- Proof complexity certainly is theoretical computer science, but has some interesting practical applications (and more than sketched in this talk)

Resolution Refutations and Pebblings

Intuition (Repeated)

From refutation of pebbling contradiction Peb_G^2 (let us fix $d = 2$) it should be possible to extract black-white pebbling of DAG G

Tentative translation: $\mathbb{C}$ should induce black pebble on v and white pebbles on W if $\mathbb{C} \cup \{w_1 \vee w_2 \mid w \in W\} \vDash v_1 \vee v_2$

$x_1 \vee x_2$
$\overline{v}_1 \vee \overline{w}_1 \vee y_1 \vee y_2$
$\overline{v}_1 \vee \overline{w}_2 \vee y_1 \vee y_2$
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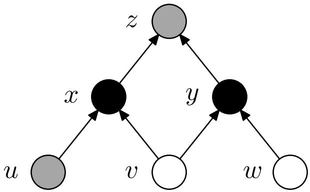

What If Refutations Misbehave?

Problem

What if a refutation doesn't feel like respecting our intuition?

Seems hard to force refutations to "follow pebbling rules"

Toy example:

$x_1 \lor x_2$
$\overline{v}_1 \lor \overline{w}_1 \lor y_1 \lor y_2$
$\overline{v}_1 \lor \overline{w}_2 \lor y_1 \lor y_2$
$\overline{v}_2 \lor \overline{w}_1 \lor y_1 \lor y_2$
$\overline{v}_2 \lor \overline{w}_2 \lor y_1 \lor y_2$

- start with previous set of clauses,
- write down some axioms for z,
- resolve over x_1, x_2, y_2,
- and erase clauses to save space

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$\overline{v}_2 \vee \overline{w}_2 \vee y_1 \vee z_1 \vee z_2$
$\overline{x}_1 \vee \overline{y}_2 \vee z_1 \vee z_2$
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$$\overline{v}_2 \lor \overline{w}_1 \lor y_1 \lor z_1 \lor z_2$$
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A Dangerous Example

If v and w true, then y or z must be true!?

Doesn't correspond to anything according to our intuition, but too dangerous to leave untranslated!

$$\overline{v}_1 \lor \overline{w}_1 \lor y_1 \lor z_1 \lor z_2$$
$$\overline{v}_1 \lor \overline{w}_2 \lor y_1 \lor z_1 \lor z_2$$
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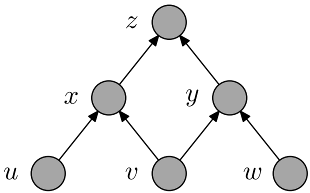

A Dangerous Example

If v and w true, then y or z must be true!?

Doesn't correspond to anything according to our intuition, but too dangerous to leave untranslated!

$\overline{v}_1 \lor \overline{w}_1 \lor y_1 \lor z_1 \lor z_2$
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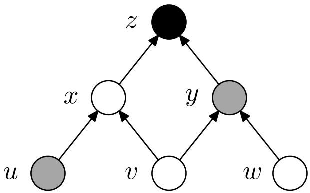

Interpret Clauses as Pebbles and “Blobs”

Solution: new pebble game with “fuzzy” black pebbles covering multiple vertices

Notation $[B]\langle W \rangle$ for

- black “blob” B with
- associated (regular) white pebbles W

“If all vertices in W is true then some vertex in B true”

Introduction move:
Black pebble on v with white pebbles on predecessors

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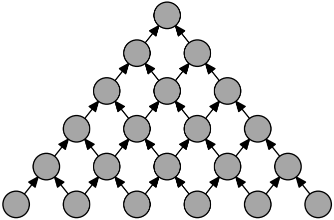

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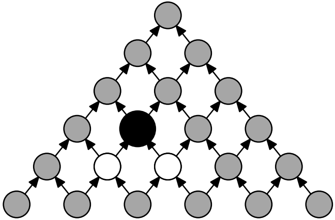
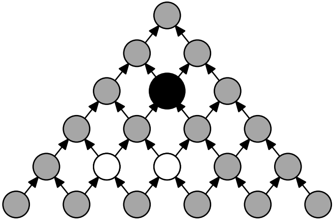


Inflations and Merger Moves

Inflation move:
Enlarge black blob and/or add white pebbles



Merger move:
Join $[B_1]\langle W_1\rangle$ & $[B_2]\langle W_2\rangle$ by removing unique common black-white vertex

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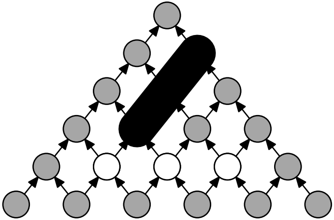

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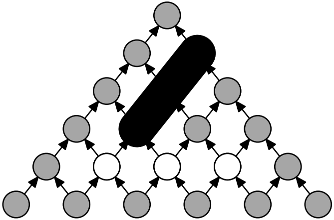

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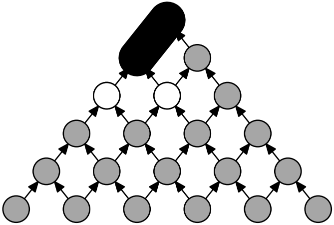

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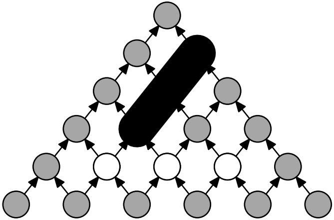

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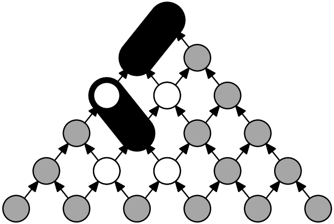

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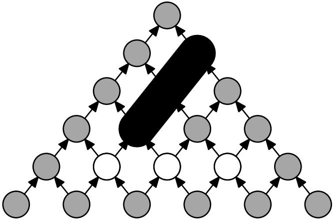

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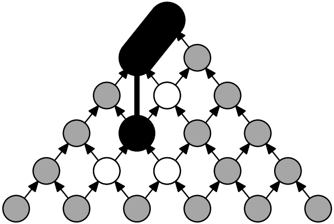

The Blob-Pebble Game in All Its Formal Glory

Blob-pebble game

Blob-pebbling of G: sequence of sets $\mathcal{P} = \{\mathbb{S}_0, \ldots, \mathbb{S}_\tau\}$ such that $\mathbb{S}_0 = \emptyset$, $\mathbb{S}_\tau = \{[z]\langle\emptyset\rangle\}$ and $\mathbb{S}_t$ is obtained from $\mathbb{S}_{t-1}$ by:

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Blob-Pebbling Price

Blob-pebbling price

- Charge for longest sequence of black blobs $B_1, \ldots, B_s$ such that $B_i \not\subseteq \bigcup_{j<i} B_j$
- For every $[B]\langle W \rangle$, charge for all white pebbles $w \in W$ that are below all $b \in B$

Blob-Peb(G) = min cost to get to $[z]\langle\emptyset\rangle$ for any pebbling of G

Example: these blobs and pebbles have cost 5

- white pebbles cost 3 (one "green" pebble not below)
- black blobs cost 2 (because of overlap)

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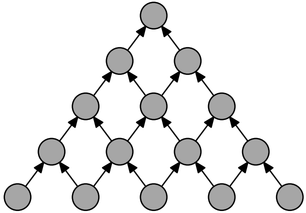

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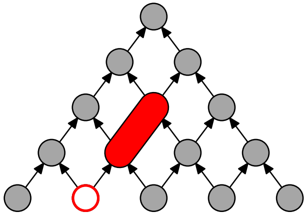

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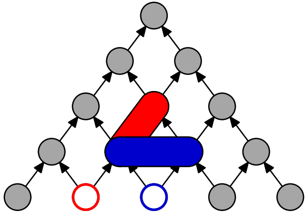

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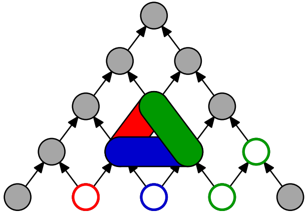

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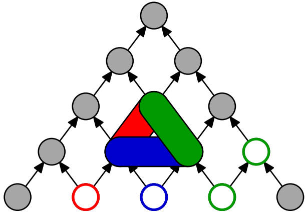

Blob-Pebbling and Space Lower Bounds

Theorem

If π is a resolution refutation of Peb^2_G then there is a blob-pebbling $\mathcal{P}_\pi$ of G such that $cost(\mathcal{P}_\pi) \leq Sp(\pi)$

Lower bounds on blob-pebbling price for G

$\Downarrow$

Lower bounds on clause space for Peb^2_G

$\Downarrow$

separation of length and space

Lower-Bounding Blob-Pebbling Price

Can analyze blob-pebblings on trees and pyramids

Theorem

If $\mathcal{P}$ is a blob-pebbling of a binary tree or pyramid of height h, then $cost(\mathcal{P}) = \Omega(h)$

More general graphs currently out of reach
(but we are working on it...)

The Key Idea for Pyramids

Define potential of set of blobs and pebbles $\mathbb{S} = \{[B_i]\langle W_i \rangle \mid i = 1, \ldots, m\}$ currently in pyramid as measure of "how good" these blobs and pebbles are

Then prove:

- Current potential of $\mathbb{S}_t$ upper-bounded by max cost so far of any $\mathbb{S}_{t'}$, $t' \leq t$
- Final pebble configuration consisting of single black blob on sink has potential $\Theta(h)$

Implies cost $\Omega(h)$ for any blob-pebbling reaching $[z]\langle\emptyset\rangle$

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Potential of Blobs and Pebbles

- $U\{\succeq j\}$: vertices in U on or above level j (sources on level 0, sink z on level h)
- measure $m(U)$ of U: $\max\{j + 2|U\{\succeq j\}| : U\{\succeq j\} \neq \emptyset\}$
- U blocks $[B]\langle W\rangle$ if $(U \cup W) \cap P \neq \emptyset$ for every path P from a source vertex such that $B \subseteq P$
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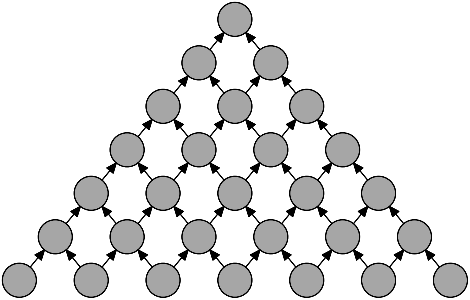
Example of Blocking Set and Potential



Block blue, green, and red blobs

Measure of blocking set is $\max\{0 + 3 \cdot 2, 1 + 2 \cdot 2\} = 6$

Which happens to be optimal, so the potential is also 6

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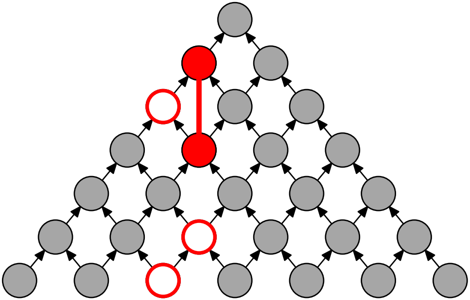

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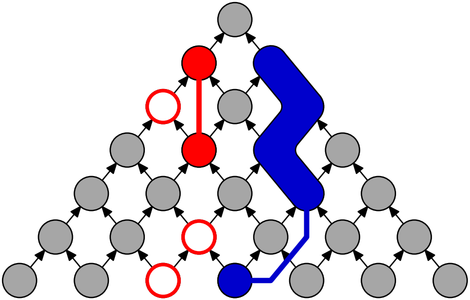

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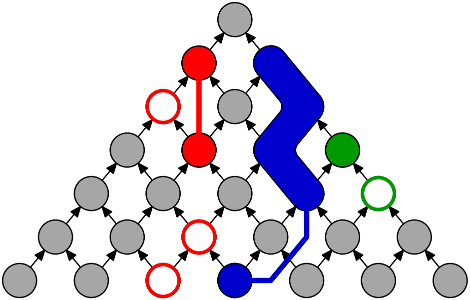

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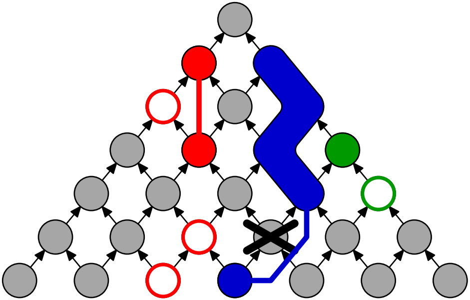

Block blue, green, and red blobs

Measure of blocking set is $\max\{0 + 3 \cdot 2, 1 + 2 \cdot 2\} = 6$

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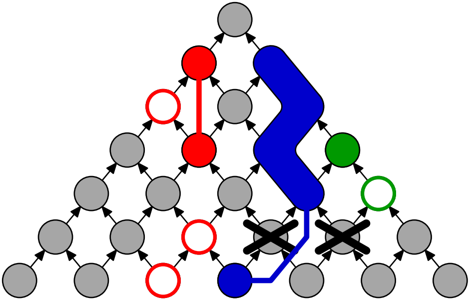

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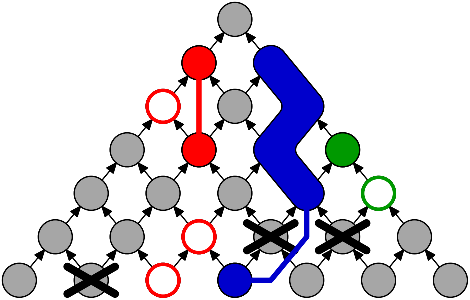

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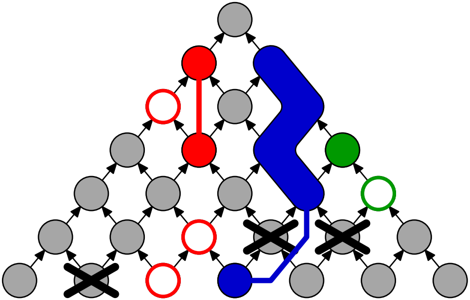

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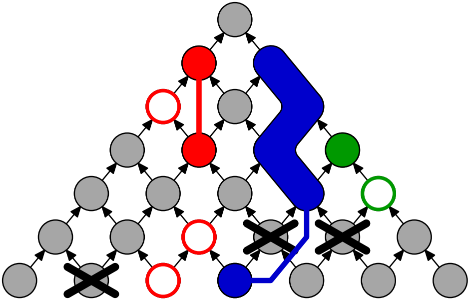

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Outline of Lower-Bound Proof

- Suppose U_t blocks $\mathbb{S}_t$ and $\mathrm{pot}(\mathbb{S}_t) = m(U_t)$
- By induction hypothesis, $\mathrm{pot}(\mathbb{S}_t) \leq C \cdot \max_{t' \leq t}\{cost(\mathbb{S}_{t'})\}$
  Want to show $\mathrm{pot}(\mathbb{S}_{t+1}) \leq C \cdot \max_{t' \leq t+1}\{cost(\mathbb{S}_{t'})\}$
- Sufficient: $\mathrm{pot}(\mathbb{S}_{t+1}) \leq \max\{\mathrm{pot}(\mathbb{S}_t), C \cdot cost(\mathbb{S}_{t+1})\}$
- Inflation, merger, erasure or introduction on non-source at time $t+1 \Rightarrow U_t$ blocks $\mathbb{S}_{t+1} \Rightarrow$ no potential increase
- Introduction on source v: now U_t may not block $\mathbb{S}_{t+1}$.
  $U_t \cup \{v\}$ blocks $\mathbb{S}_{t+1}$, but ind. hyp. doesn't provide enough info to show $m(U_t \cup \{v\}) \leq \max\{\mathrm{pot}(\mathbb{S}_t), C \cdot cost(\mathbb{S}_{t+1})\}$!
- Needed property: $\exists$ constant C' s.t. for any $\mathbb{S}$ there is U blocking $\mathbb{S}$ with $\mathrm{pot}(\mathbb{S}) = m(U)$ and $|U| \leq C' \cdot cost(\mathbb{S})$
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