DD2445 COMPLEXITY THEORY: LECTURE 11

LAST TIME

- Wrapped up circuit complexity (for now)
- Started talking about RANDOMIZED COMPUTATION

Probabilistic Turing machine (PTM)
- Two transition functions in every state
- Flips coin which one to take
- Output $M(x)$ random variable

For language L, let $L(x) = \begin{cases} 0 & \text{if } x \notin L \\ 1 & \text{if } x \in L \end{cases}$

BPP Bounded-error probabilistic polynomial time

Class of languages for which $\exists$ PTM M that always runs in polynomial time (regardless of coinflips) and $\forall x$

$$\Pr\left[M(x) = L(x) \right] \geq \frac{2}{3}$$

Probability not over input x
only over internal randomness of algorithm

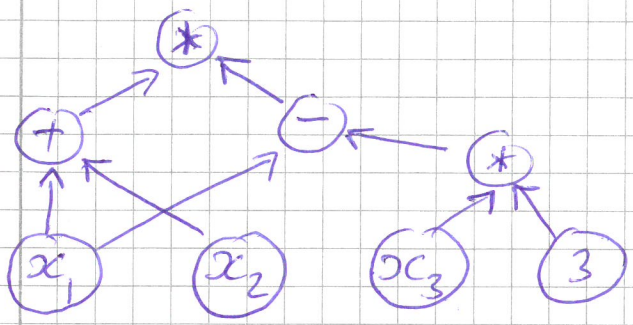
$$P \subseteq BPP \subseteq EXP$$

Believe $P = BPP$
But there are things we currently don't know how to do efficiently without randomness

POLYNOMIAL IDENTITY TESTING

Input Algebraic circuit representing multivariate polynomial with integer coefficients

Task: Decide whether polynomial identically zero



$$(x_1 + x_2)(x_1 - 3x_3)$$
$$= x_1^2 - 3x_1x_3 + x_1x_2 - 3x_2x_3$$

$$ZEROP = \left\{ \text{algebraic circuits corresponding to identically zero polynomials} \right\}$$

Idea for randomized algorithm

- Pick random input $(\alpha_1, \alpha_2, \ldots, \alpha_n)$
- Set $x_i = \alpha_i$ for $i = 1, \ldots, n$ and evaluate circuit
- If result $\neq 0$, declare circuit $\notin$ ZEROP
- If result $= 0$, say probably $\in$ ZEROP

Always correct on no-instances

But what about yes-instances?

SCHWARTZ-ZIPPEL LEMMA

Let $p(x_1, \ldots, x_m)$ non-zero polynomial of total degree $\leq d$. Let S finite set of integers. Then for $a_1, \ldots, a_m$ chosen from S uniformly randomly with replacements it holds that

$$\Pr\left[p(a_1, \ldots, a_m) \neq 0 \right] \geq 1 - \frac{d}{|S|}$$

Proof By induction over m

Base case $m = 1$: Univariate polynomial of degree $\leq d \Rightarrow$ at most d distinct roots

Induction step: Write

$$p(x_1, x_2, \ldots, x_m) = \sum_{i=0}^{d} x_1^i \, p_i(x_2, \ldots, x_m)$$

p is non-zero $\Rightarrow$ some p_i non-zero.
Pick largest i s.t. p_i non-zero.
By induction hypothesis

$$\Pr_{a_2, \ldots, a_m}\left[p_i(a_2, \ldots, a_m) \neq 0 \right] \geq 1 - \frac{d-i}{|S|}$$

When $p_i(a_2, \ldots, a_m) \neq 0$ we get that

$$p(x_1, a_2, \ldots, a_m) = \sum_{j=0}^{i} x_1^j \, p_i(a_2, \ldots, a_m)$$

is a univariate polynomial of degree $\leq i$, and so $= 0$ for at most i values, so

$$\Pr\left[p(a_1, a_2, \ldots, a_m) \neq 0 \right] \geq \left(1 - \frac{i}{|S|}\right)\left(1 - \frac{d-i}{|S|}\right) \geq 1 - \frac{d}{|S|}$$

RV

Refined testing idea

Circuit of size $m \Rightarrow \leq m$ multiplications

$\Rightarrow$ degree $\leq 2^m$

Let $S = \{1, 2, \ldots, 10 \cdot 2^m\}$

Pick $a_i \in S$ randomly & evaluate circuit

By Schwartz-Zippel, if circuit encodes non-zero polynomial, then 90% chance of seeing non-zero output

If circuit encodes zero polynomial, then output always zero

Done? Not quite...

Problem: If degree $\approx 2^m$, then numbers grow as large as $(10 \cdot 2^m)^{2^m} \Rightarrow$ exponentially many bits

Hard to achieve in polynomial time...

Solution: "FINGERPRINTING"

Compute modulo some $k \in [2^{2m}]$

Computing modulo k
After each operation, divide by k and take remainder

Suppose $y = C(a_1, \ldots, a_m)$

If $y = 0$, then $y = 0 \pmod k$

Claim 5
If $y \neq 0$, then randomly chosen $k \in [2^{2m}]$ will not divide y with prob $\geq \delta = \frac{1}{4m}$

Given this claim, run test $O(m)$ times and accept only if always get 0 output
$\Rightarrow$ arbitrarily high constant probability of success

Proof of Claim 5

Assume $y \neq 0$. $y \leq (10 \cdot 2^m)^{2^m}$

Let B = prime factors of y.
Sufficient to show that with prob $\geq \delta$ k is a prime not in B

y has at most $\log y \leq 5m\,2^m$ prime factors

By Prime Number Theorem (*) — constant is actually 1
$\#\text{primes} \leq N \sim N / \ln N$

$\#\text{primes} \leq 2^{2m} \sim \frac{2^{2m}}{2m} > \frac{2^{2m}}{4m}$ for large enough m

$5m\,2^m = o\left(\frac{2^{2m}}{2m}\right) < \frac{2^{2m}}{8m}$ for large enough m

$$\Pr[k \text{ prime not in } B] \geq \frac{2^{2m}/8m}{2^{2m}} = \frac{1}{8m}$$

*) See Thm A.23 in Arora-Barak for sufficient, simpler version

Many natural randomized algorithms have one-sided error

Might make mistake when $x \in L$ but never when $x \notin L$ or the other way around (we just saw one such example)

DEF 6 $RTIME(T(n))$ contains every language L for which $\exists$ PTM M running in time $O(T(n))$ such that

$$x \in L \Rightarrow Pr[M(x) = 1] \geq 2/3$$
$$x \notin L \Rightarrow Pr[M(x) = 0] = 1$$

$$RP = \bigcup_{c \in \mathbb{N}^+} RTIME(n^c)$$

OBS 7 $RP \subseteq NP$

Pf Every accepting branch is a certificate.

Don't know if $BPP \subseteq NP$.

RP: "Never false positives" (positive answers always right)

$coRP = \{L \mid \bar{L} \in RP\}$ "Never false negatives"

Given general PTM M, can define random variable

$T_{M,x}$ = running time of M on x.

Take expectation of this random variable

$$E[T_{M,x}] = \sum_{t=1}^{\infty} t \cdot Pr[T_{M,x} = t]$$

Say M has expected running time $T(n)$ if

$$\forall x \in \{0,1\}^* \quad E[T_{M,x}] \leq T(|x|)$$

DEF 8 $ZTIME(T(n))$ contains all languages L for which $\exists$ PTM M that runs in expected time $O(T(n))$ such that

$$Pr[\ M(x) = L(x) \mid M \text{ halts}\ ] = 1$$

$$ZPP = \bigcup_{c \in \mathbb{N}^+} ZTIME(n^c)$$

"Zero-sided error"

ZPP zero-error probabilistic polynomial time

THM 9 $ZPP = RP \cap coRP$

Proof Exercise.

Also immediately clear from def

$$RP \subseteq BPP$$
$$coRP \subseteq BPP$$

ROBUSTNESS OF DEFINITIONS

(a) Error probability: constant 2/3 arbitrary
(b) Can use expected running time instead of worst case
(c) can use biased coins
(d) Can even use imperfect random sources ("weak random sources")

Will show (a) — see Sec 7.4 for the rest

LEMMA 10 For $c>0$ constant, let $BPP_{1/2+n^{-c}}$ denote class of languages L for which $\exists$ poly-time PTM M s.t. $\forall x \in \{0,1\}^*$

$$\Pr[M(x)=L(x)] \geq \frac{1}{2} + |x|^{-c}$$

Then $BPP_{1/2+n^{-c}} = BPP$.

Need to show: can go from success prob $\frac{1}{2}+n^{-c}$ to $2/3$.

Show sth stronger: can go to $1-2^{-n^d}$ exponentially small failure prob.

THM 11 (ERROR REDUCTION FOR BPP)

Suppose $\exists$ poly-time PTM M for L s.t.

$$\forall x \quad \Pr[M(x)=L(x)] \geq 1/2 + |x|^{-c}.$$

Then $\forall d>0$ $\exists$ poly-time PTM M' s.t.

$$\forall x \quad \Pr[M(x)=L(x)] \geq 1 - 2^{-|x|^d}$$

Proof

M' runs M for $k = 8|x|^{2c+d}$ times, collects answers, and takes majority vote.

How confident can we be that this is correct?
Use material from App A.2.1 & A.2.4

Let $X_i = \begin{cases} 1 & \text{if } i\text{th run of } M \text{ gets } x \text{ right} \\ 0 & \text{otherwise} \end{cases}$

$\Pr[X_i = 1] = p$ for $p \geq \frac{1}{2} + |x|^{-c}$ [suppose $p = 1/2 + |x|^{-c}$ for simplicity]

$$E\left[\sum_{i=1}^{k} X_i\right] = kp = \underbrace{\frac{8|x|^{2c+d}}{2}}_{\text{half}} + \underbrace{8|x|^{c+d}}_{\text{margin}}$$

"If you repeat independent trials sufficiently many times, you will get very close to expected value with very high probability"

LEMMA 12 (CHERNOFF BOUND) — deviation from expected value

$$\Pr\left[\left|\sum_{i=1}^{k} X_i - pk\right| > \delta pk\right] < \exp\left(-\frac{\delta^2}{4} pk\right)$$

Plug in $p = \frac{1}{2} + |x|^{-c}$

$\delta = |x|^{-c}/2$

We will be correct unless $\sum_{i=1}^{k} X_i < pk - \delta pk$.
That probability is bounded by

$$\exp\left(-\frac{1}{4|x|^{2c}} \cdot \frac{8|x|^{2c+d}}{2}\right) = \exp\left(-|x|^d\right) < 2^{-|x|^d}$$ [4]

Relationship between BPP and other classes?

THM 12 $BPP \subseteq P/poly$

THM 13 $BPP \subseteq \Sigma_2^P \cap \Pi_2^P$ $(\subseteq PH)$

Both proofs use error reduction in Thm 11 plus some other ideas.

Proof of Thm 13 is extremely neat...
But will have to skip it due to time constraints.

Try to sketch proof of Thm 12

Proof of Thm 12

If $L \in BPP$, then by Thm 11 (and Prop 3)
$\exists$ PTM M that on input size n

- uses m random bits
- gets answer right except with prob $2^{-(n+1)}$

Let r be the random bits

Say r bad for x if $M(x,r) \neq L(x)$

For every x, M succeeds with prob $\geq 1 - 2^{-(n+1)}$
$\Rightarrow$ out of 2^m random strings, $\leq 2^m / 2^{n+1}$ bad for x.

$$\left| \{ r \mid r \text{ bad for some } x \} \right| \leq \sum_{x \in \{0,1\}^n} \left| \{ r \mid r \text{ bad for this } x \} \right|$$

$$\leq \frac{1}{2} 2^n \cdot \frac{2^m}{2^{n+1}} = 2^m / 2$$

But this means that there is at least one random string $r^* \in \{0,1\}^m$ (infact, at least half) that are good for all $x \in \{0,1\}^n$

Run M with this r^* as advice!

Checking Thm 11 again, r^* will have poly size.

What about complete problems for BPP?

Typical complete problems

∃ TM of correct type running with resource bound such-and-such

"correct type": DTM – easy to check
NDTM – easy to check
} syntactic

BPP-style: accept x with prob ≥ 2/3
or prob ≤ 1/3
but not in between

Undecidable to check

Hierarchy theorems?

Fail for similar reasons.

A final useful notion: Randomized reductions

DEF 14 Language B reduces to language C under randomized reductions, denoted $B \leq_r C$, if ∃ PTM M s.t.

$$\forall x \in \{0,1\}^* \quad Pr\left[C(M(x)) = B(x) \right] \geq 2/3$$

Not transitive

But if $C \in BPP$ and $B \leq_r C$ then $B \in BPP$

Could have defined NP in terms of randomized reductions instead (if BPP better formalization of "efficient computation")

R XII

$$NP = \{ L \mid L \leq_p 3\text{-SAT} \}$$

DEF 15

$$BP \cdot NP = \{ L \mid L \leq_r 3\text{-SAT} \}$$