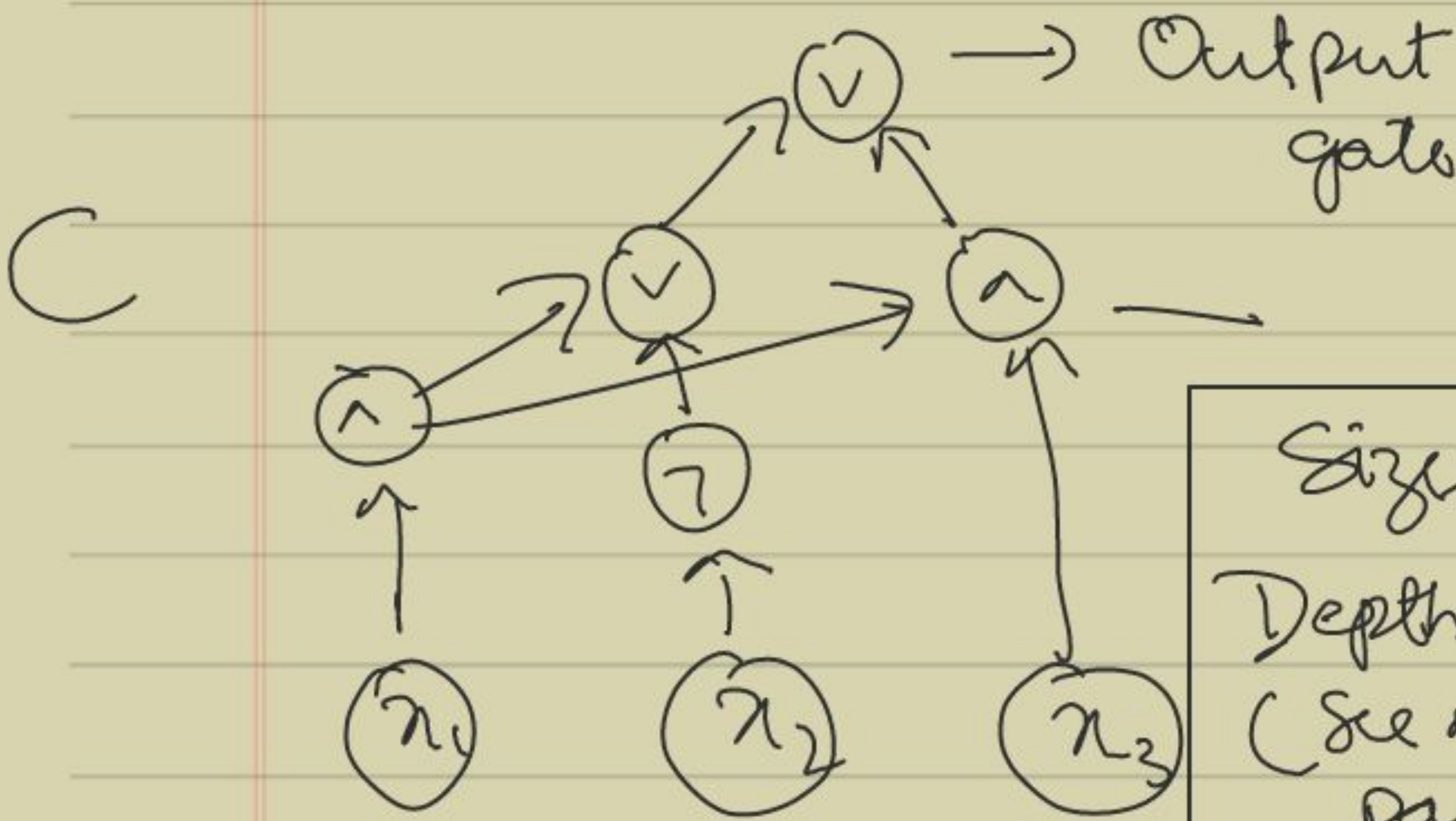
Boolean Circuits & P/poly

Want to show: SAT has no poly-time algorithms.

→ SAT seems to be hard at each input length. Why not try to understand the most efficient way to solve SAT at each input length n & show that this running time is not a polynomial function of n.?

→ Leads to computational models that work with inputs of a fixed length.

Boolean circuits



Size = 8
Depth = 3
(See next page)

- → Directed acyclic graphs (DAG)
- → Sources labelled by variables.
- → Internal node labelled by $\neg, \wedge, \vee$ & have either 1, 2, or 2 in-neighbours respectively. Called "gates".
- → One marked output gate (can also have more)
- → Computes $f : \{0,1\}^{n = \# \text{variables}} \longrightarrow \{0,1\}$

Compare with Boolean formulas:

→ Circuits are more general

→ A formula corresponds to a directed tree.

We think of a circuit as an algorithm computing a function on inputs of a fixed length.

Complexity of algorithm measured by

1. Size = # of vertices. (analogous to running time)
2. Depth = length of longest path from variable to output.

To talk about circuits for a language, we need one for each input length.

$\{C_n\}_{n\in\mathbb{N}}$ - family of circuits

(C_n depends on n inputs).

Say $\{C_n\}_{n\in\mathbb{N}}$ has size $T(n)$ if

$|C_n| \leq T(n)$ for each n.

$\{C_n\}_{n\in\mathbb{N}}$ decides a language $L \subseteq \{0,1\}^*$

if for any $x \in \{0,1\}^n$

$$x \in L \iff C_n(x) = 1.$$

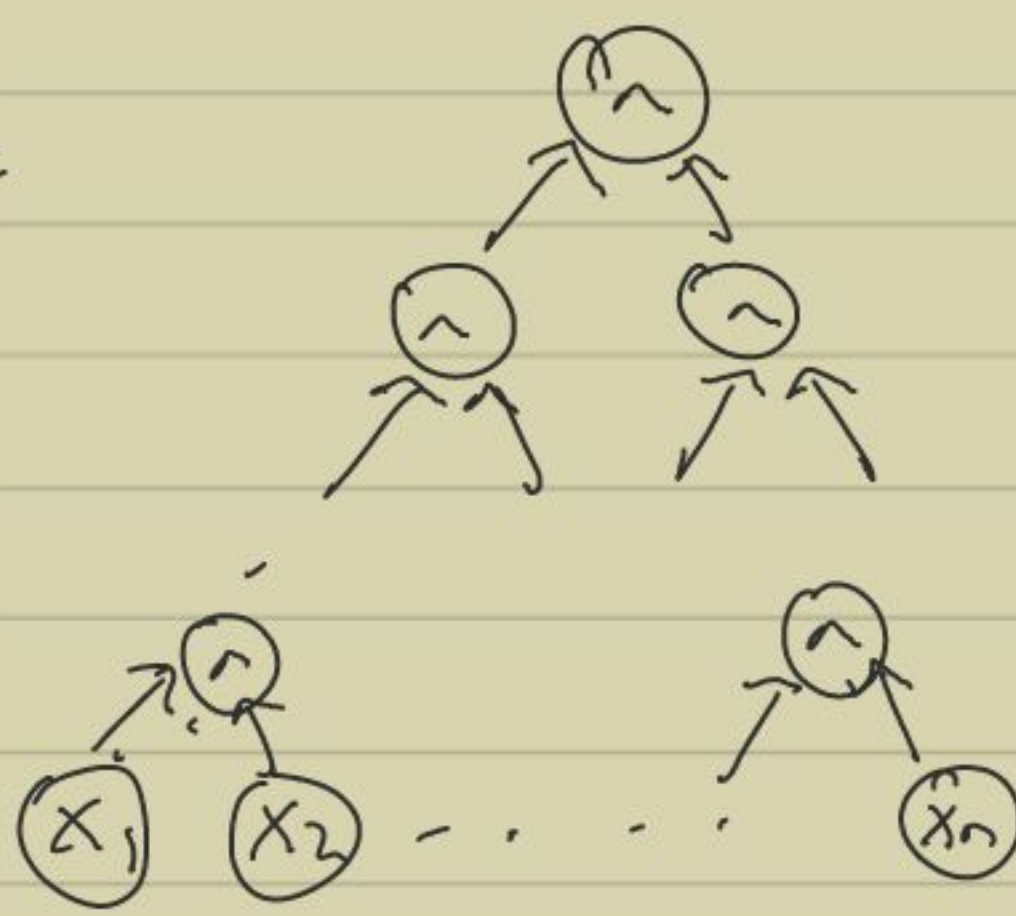
$\mathrm{SIZE}(T(n)) = \{L \mid L$ decided by a circuit family of size $T(n)\}$.

$$P/poly = \bigcup_{c} SIZE(n^c).$$

(i.e languages decided by a polynomial-sized circuit family)

Eg: $L = \{1^n \mid n \in \mathbb{N}\} \in P/poly$

$C_n =$



More generally, any unary language.

(see next page)

Another definition: TMs with advice

DTM augmented with an "advice string" that depends on the length of the input

Eg: L a unary language i.e

$$L \subseteq \{1^n \mid n \in \mathbb{N}\}$$

So at each input length, L contains either 0 or 1 string. Hence, given one bit of advice (does $1^n \in L$?) a DTM M can decide L in polynomial-time.

$L \in \mathrm{DTIME}(T(n))/a(n)$

(decidable by $T(n)$-time TMs with $a(n)$ bits of advice)

if there exist $y_n \in \{0,1\}^{a(n)}$ for each n s.t.

$x \in L \iff M(x, y_n) = 1$

where M runs in time $T(n)$.

L a unary language $\Rightarrow$

$L \in \mathrm{DTIME}(O(n))/1$

Even includes some undecidable languages!

[Eg: Unary-Halt = $\{1^n \mid n\text{th TM halts on empty input}\}$]

Thm 1: $P/\text{poly} = \bigcup_{c,d} \text{DTIME}(n^c)/n^d$.

Pf: ($\Rightarrow$) Assume $L \in P/\text{poly}$

Then L is decided by a circuit family $\{C_n\}_{n \in \mathbb{N}}$ where $|C_n| \leq \text{poly}(n)$

Then we can also decide L with $\text{poly}(n)$ bits of advice "encoding" the circuit C_n

describe the circuit in a reasonable way.

The machine M on input (x, y_n) just runs C_n on input x, which can be done in polynomial time

($\Leftarrow$) Say L decided by poly-time DTM M with poly(n) bits of advice.

Want: a circuit C_n on inputs of length n.

Idea: Go back to proof of Cook-Levin.

We can assume M is an oblivious k-tape TM. On input (x, y_n) ↳ advice the machine M produces a sequence of snapshots (current state & symbol being read)

$z_1, \ldots, z_{poly(n)}$.

Each z_i is a constant number of bits & can be computed from

$z_{i_1}, \ldots, z_{i_k}$ where
$i_1, \ldots, i_k < i$ are the previous time-steps where M scanned the same location of the tape in the k tapes.

The dependence of z_i on $z_{i_1}, \ldots, z_{i_k}$ is determined by the rules of M & we can write an $O(1)$-sized circuit that implements these rules.

Thus, we can construct a circuit C_n that reconstructs all the snap-shots & accepts if & only if the final snapshot is accepting. $\square$

Corollary2: $P \subseteq P/poly$.

In fact, if $L \in P$, then L is decidable by a circuit family $\{C_n\}_{n \in \mathbb{N}}$ where C_n can be constructed by an algorithm in poly(n) time. (The above proof shows this. The entire proof is algorithmic, except for the constr-uction of the advice string y_n.)

$\rightarrow$ Such circuit families are called P-uniform

This gives us a new approach to

P vs NP.

Show that some problem in NP does not have polynomial-sized circuits

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Is this feasible?

After all, P/poly contains even some undecidable languages!

But we believe it is true that

$NP \not\subseteq P/poly$ because...

Thm 2 [Karp-Lipton thm]:

If $NP \subseteq P/poly$, then $PH = \Sigma_2^p$

Proof: We will show that if
$NP \subseteq P/poly$, then $\Pi_2^P = \Sigma_2^P$

Sufficient to show: $\Pi_2^P \subseteq \Sigma_2^P$
(ex.)

Say $L \in \Pi_2^P$. There is a poly-time DTM M s.t.

$$x \in L \iff \forall y_1 \exists y_2 \; M(x, y_1, y_2) = 1$$

strings of length poly(n)

Define:

$$L' = \{ (x, y_1) \mid \exists y_2 \; M(x, y_1, y_2) = 1 \}$$

Obs: (1) $L' \in NP$

(2) $L = \{ x \mid \forall y_1 \; (x, y_1) \in L' \}$

Since $L' \in NP$, any instance of L' can be the reduced in poly-time to an instance of SAT of length $m = poly(n)$

Idea 1: Use the first certificate of the Σ_2^P algorithm to get a circuit that solves SAT on inputs of length m.

DTM M' to show $L \in \Sigma_2^P$:

certificate hopefully encoding C solving SAT on inputs of length m

$M'(x, y_0, y_1)$:

(1) Reduce $(x, y_1) \overset{?}{\in} L'$ to checking $\varphi \overset{?}{\in} SAT$

(2) Check that the circuit C outputs 1 on φ. If so accept & o/w reject.

Problems: What if y_0 is not a circuit solving SAT correctly?

Eg: y_0 encodes a circuit C that accepts everything! Then we also accept $x \notin L$.

Fix: Use y_0 to get a circuit C that outputs a satisfying assignment of a satisfiable CNF.

Ex: If $NP \subseteq P/poly$, then there is a multi-output poly-sized circuit family that outputs a satisfying assignment of any satisfiable CNF φ.

With the fix, we can no longer be fooled into accepting when we should reject.

So final (correct) version of M':

$M'(x, y_0, y_1)$

(1) Reduce $(x, y_1) \overset{?}{\in} L'$ to checking $\varphi \overset{?}{\in} SAT$

(2) Check that circuit C encoded by y_0 outputs a satisfying assignment of φ. If so, accept & otherwise reject.

Quick proof of correctness:

$x \in L \Rightarrow \forall y_1\ (x, y_1) \in L'$

$\Rightarrow \exists y_0\ \forall y_1\ M'(x, y_0, y_1) = 1$

the "correct" circuit C

Conversely,

$x \notin L \Rightarrow \exists y_1\ (x, y_1) \notin L'$

$\Rightarrow \exists y_1\ \varphi$ is not sat-
-isfiable

$\Rightarrow \forall y_0\ \exists y_1\ M'(x, y_0, y_1) = 0$

(because M' does not get a satisfying assignment to φ).